Astronomy 596/496 NPA Spring 2005
Final Problem Set

Due at or before 5 pm, Monday December 14: *No extensions!*

This problem set is open book, open notes, and open web, but you are not to consult with anyone else.

1. *Gamma-Rays from Cosmic Rays.*

   (a) Make an estimate of the radius of the Galactic disk, and of our distance from the center. Then use the equation of γ-ray radiation transfer to predict the ratio of the γ-ray flux toward the Galactic center ($\ell = 0$) to the flux towards the anticenter ($\ell = 180°$), both measured along the Galactic plane ($b = 0$). Figure 2c of Hunter, S.D., et al. 1997, ApJ, 481, 205 shows $\phi_\gamma(\ell, b)$ for ranges in Galactic latitude and longitude at 300–1000 MeV. How well do the data from the central band ($b < 2°$) agree with your prediction?

   (b) Find the minimum energy (i.e., the threshold) for a cosmic ray proton to produce a pion via collisions with ISM hydrogen: $pp \rightarrow pp\pi^0$.

   (c) With the advent of CGRO a longstanding debate regarding cosmic ray origins was put to rest. It was argued in class that most cosmic rays (i.e., certainly those below $\sim$ 100 TeV) are Galactic in origin. However, this point was not always widely accepted, and into the 1960's the possibility of an extragalactic (also called "metagalactic") cosmic-ray origin was considered viable. We can test this hypothesis by calculating the γ-ray flux from the Small Magellanic Cloud (SMC) in this model. To do this, assume that the cosmic ray flux measured at earth reflects a single universal (metagalactic) value. The $pp \rightarrow pp\pi^0$ cross section is in fact a strong function of energy, but for now use a mean value of $\sigma_{pp}^{\gamma} = 8$ mb, appropriate for cosmic ray energies above the value shown in part (b); note that for each π^0, two γ-rays are created. The SMC gas mass is measured to be $M_{\rm SMC} = 6.4 \times 10^8 M_\odot$, and its distance is 60 kpc. Using this information, compute the expected γ-ray flux from the SMC, assuming it is effectively a point source given the resolution of the CGRO.

   (d) Compare your prediction in (c) with the observed upper limit on the SMC γ-ray flux, $\Phi_\gamma < 4 \times 10^{-8}$ photons cm^{-2} s^{-1} (Lin, Y.C., et al. 1996, ApJS, 105, 331). Can you rule out a metagalactic origin for cosmic rays?

2. *Neutrino Oscillations.* Observations of solar neutrinos are consistent with $\nu_e - \nu_x$ oscillations which are best fit by the LMA solution having $\Delta m_{21}^2 \simeq 8 \times 10^{-5}$ eV^2, and $\tan^2\theta \simeq 0.47$. This solution has potential implications for observations of atmospheric neutrinos. For the following, assume a two-species solution is appropriate.

   (a) For atmospheric ν_e with energy $E_\nu = 1$ GeV, what is the vacuum oscillation length $L_{\rm max}$ at which the ν_e disappearance is maximum?

   (b) What is the maximum distance actually traveled by atmospheric neutrinos? (Note that $R_\oplus = 6400$ km.) For 1 GeV atmospheric ν_e which travel this distance, find the fraction which "disappear," given the favored LMA solution.

(c) The situation is in fact more complicated than suggested by part (b), since we are dealing with ν_e oscillations and thus, as for the Sun, we are obliged to consider matter effects as well. Use the resonance condition for MSW oscillations in matter to find the required density $\rho_{\rm crit} = m_u n_e^{\rm crit} \simeq 2\rho$ for this resonance condition to hold for a 1 GeV ν_e. If this density is obtained within the earth, then the MSW effect can be important and must be included in a full analysis.

3. *SN 1987A and Neutrino Mass.* Using the SN 1987A neutrino signal–in particular, its time and energy dispersion–one can infer an interesting limit on the neutrino mass.

(a) Let us assume that the neutrinos are ultrarelativistic; we will verify the consistency of this assumption. Given this, find an expression for the neutrino speed v in terms of its (total) energy E and rest mass $m \ll E$.

(b) Use the result from (a) to show that for one neutrino detection event, the (lab frame) elapsed time between emission and observation is

$$t_{\rm obs} - t_{\rm em} \simeq t_0 \left(1 + \frac{m^2}{2E^2}\right) \tag{1}$$

where $t_0 = D_{\rm LMC}/c$ is the light travel time. Using this, show that for two neutrino events, with energies E_1 and E_2, their *differences* in travel times are

$$\Delta t_{\rm obs} - \Delta t_{\rm em} \simeq \frac{t_0 m^2}{2} \left(\frac{1}{E_1^2} - \frac{1}{E_2^2}\right) \tag{2}$$

where $\Delta t_{\rm obs}$ is the time between detections and $\Delta t_{\rm em}$ is the time between emission.

(c) The observed spread of neutrino arrival times is $\Delta t_{\rm obs} \simeq 12$ s. Assume for simplicity (but not realistically!) that all of the neutrinos are emitted simultaneously: $\Delta t_{\rm em} = 0$. The detected energy range goes from about 8 to 40 MeV. If we attribute all of the time structure as due to neutrino mass effects (i.e., all ν_e's have the same mass, but the faster ones arrive first), use these data to infer a limit on the neutrino mass. You make take $D_{\rm LMC} = 50$ kpc. Was the assumption of ultrarelativistic neutrinos justified? How does your limit compare with laboratory limits on the ν_e mass?

4. *Neutrino Oscillations and Nonbaryonic Dark Matter.* Although we do not know neutrino masses, we have enough information from various laboratory experiments to make an important statement about the cosmic neutrino density. For simplicity, we will assume for this problem that $m_1 < m_2 < m_3$; this is not guaranteed but is the same pattern the quark and lepton masses follow and thus is known as the "normal hierarchy."

(a) As we have discussed, oscillation experiments measure only the magnitude of the mass-square difference between neutrino species, i.e., solar neutrinos and related laboratory experiments measure $\Delta m_{12}^2 = m_2^2 - m_1^2$, and atmospheric neutrinos measure $\Delta m_{23}^2 = m_3^2 - m_2^2$. Show why we do not need a new experiment to independently measure Δm_{13}^2. Given current values of $\Delta m_{12}^2 = 8 \times 10^{-5}$ eV^2 and $\Delta m_{23}^2 = 2 \times 10^{-3}$ eV^2, find the value of Δm_{13}^2.

(b) Imagine that we knew the value of one neutrino mass, say m_1, and all of the Δm^2. Show that in this case, you can recover the values of m_2 and m_3. That is, give expressions for m_2 and m_3 that depend only on m_1 and the Δm^2.

(c) Explain why the values of the Δm^2 cannot give us *any* of the mass m_1, m_2, m_3. To do this, show that in your result in part (b), if m_1 is *not* already fixed, then we can only set a *lower limit* on the other neutrino masses. Find expression for the lower limits on m_2 and m_3. Finally, combine these to find an expression for a resulting lower limit $m_{\nu,\mathrm{tot}}^{\mathrm{min}}$ on the sum $m_{\nu,\mathrm{tot}} = \sum_i m_i$ of the neutrino masses.

(d) Laboratory results from nuclear decays set limits on the combination of mass states which make up the ν_e flavor state; for simplicity let us oversimplify and take these resutls to set the *upper* limit $m_1^{\mathrm{max}} < 1$ eV.

Given m_1^{max}, and your result from part (b), compute an *upper* limit to m_2 and m_3, and an upper limit $m_{\nu,\mathrm{tot}}^{\mathrm{max}}$ to the sum of the neutrino masses.

(e) You showed in Problem Set 4 that the cosmic neutrino mass density parameter is given by

$$\Omega_\nu = \frac{\sum_i m_{\nu,i}}{46 \text{ eV}} = \frac{m_{\nu,\mathrm{tot}}}{46 \text{ eV}} \tag{3}$$

Using your results from parts (c) and (d), compute both a *lower* and *upper* limit to Ω_ν.

(f) You should have found that your upper limit gives $\Omega_\nu \ll \Omega_\mathrm{m}$, i.e., the neutrino density is much less than the matter density. Briefly explain the importance of this result for particle physics and for cosmology.