Astronomy 596/496 NPA   Fall 2009
Problem Set #1

Due in class: Friday, Sept. 11

Note: Your homework solutions should be legible and include all calculations, diagrams, and explanations. The instructor is not responsible for deciphering unreadable or illegible problem sets! Also, homework is graded on the method of solution, not just the final answer; you may not get any credit if you just state the final answer!

1. *Abundance Measures.* There is a whole thicket of abundance measures that one finds in the literature, which can unfortunately lead to confusion. The following are some exercises to get you used to the different notations and to translating among them.

   (a) Given the mass density ρ_i of nuclide species i, what is n_i, the number density in i? Express this both with and without including the nuclear binding energy of i. Show that ignoring the binding energy introduces a $< 1\%$ error in n_i. Of course, neutral matter will also contain electrons; show that neglecting the electron contribution to the mass densities also gives a $< 1\%$ error–thus, we can take ρ to be the total matter density as well as the baryonic density. Using these approximations, express ρ_i in terms of n_i and A_i, the mass number.

   (b) One way to define an abundance is the *mass fraction* $X_i \equiv \rho_i/\rho$, where $\rho = \sum_j \rho_j$ is the total baryonic mas density summed over all species. Show that X_i is unaffected by a bulk, chemically homogeneous expansion of the gas, and that the mass fractions obey $\sum_i X_i = 1$ (this also means that any given $0 \leq X_i \leq 1$, i.e., that these are really fraction).

   (c) Another useful measure is the *mole fraction* $Y_i = n_i/n_{\rm B}$. What is the relationship between Y_i and X_i? That is, find an expression for X_i in terms of Y_i and physical/nuclear constants. Show that $\sum_i Y_i \neq 1$ in general. In general, is $\sum_i Y_i \leq 1$ or ≥ 1? Prove your answer.

   (d) It is sometimes useful to define the *electron fraction* $Y_e = n_e/n_B$, where n_e is the number density of electrons, with each nuclide species i contributing Z_i electrons (assuming electrical neutrality). Find an expression for Y_e in terms of the set of Y_i. What are the upper and lower limits to Y_e? What astrophysical systems have compositions for which Y_e approaches these limits? Estimate Y_e for the Earth, assuming that nuclei of heavy elements typically have half protons and half neutrons.

   (e) Mass fractions are not in practice easy to measure directly. A more useful observable is $y_i = \mathcal{A}_i/{\rm H} = n_i/n_{\rm H}$. This is usually what observers mean when they say "abundance." Like X_i, this is dimensionless, but the "normalization" condition does not hold (show this). What is the connection between $\mathcal{A}_i/{\rm H}$ and X_i? That is, express X_i in terms of the set of $\{y_i\} \equiv \{\mathcal{A}_i/{\rm H}\}$, and express y_i in terms of X_i and X_H. Explain why mass fractions are seldom reported by observers.

   (f) Geologists and cosmochemists of course use different abundance scales, often setting Si to some arbitrary fiducial value like 10^2 or 10^6. Using your handy table of abundances from Anders & Grevesse (1989 Geochim. et Cosmochim.

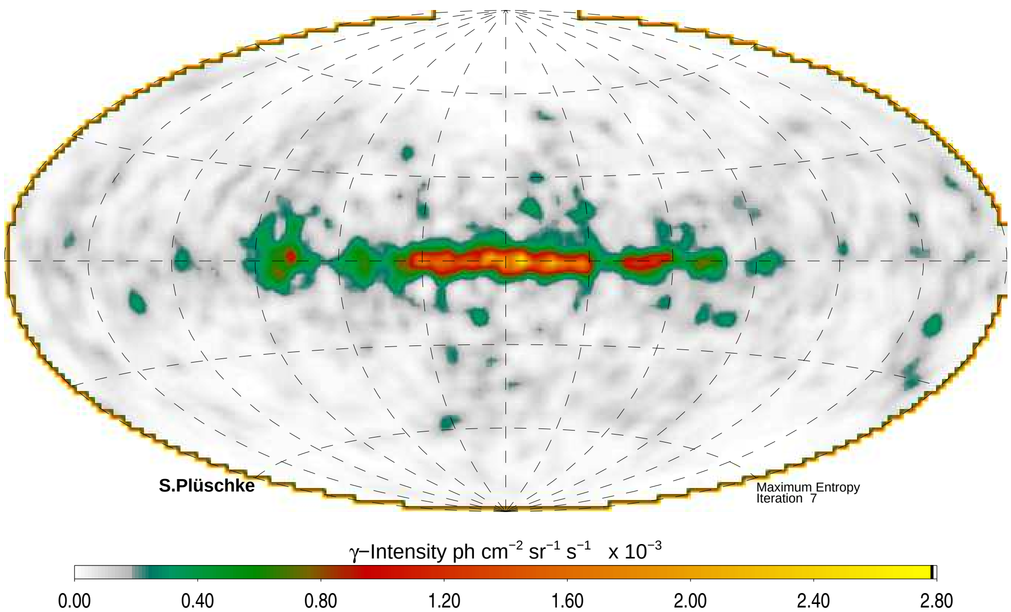
Acta) compute the solar values of: D/H, He/H, O/H, and La/H. Using the table and $Z_\odot \simeq 0.02$, what is the inferred $Y_\odot$?

2. *Nuclear Binding Energy.*

   (a) Find an expression for the binding energy B_i of nuclide i in terms of its mass defect Δ_i (and possibly Δ_p and/or Δ_n). Also find an expression for the Q value of a nuclear reaction $A(b,c)D$ in terms of the mass defects of the particles.

   (b) On the web (e.g., the Nuclear Wallet Card page on the course `links`) you will find a veritable panoply of nuclear data, including mass defect information. Use this to compute the binding energy, and B/A, of deuterium d, ^{4}He, ^{12}C, and ^{56}Fe, and ^{238}U. Find the Q value for $^3\mathrm{He}(\alpha,\gamma)^7\mathrm{Be}$.

3. *Gamma-Ray Line Astronomy.* As discussed in lcass, the isotope ^{26}Al is radioactive but long-lived, with $t_{1/2}(^{26}\mathrm{Al}) = 0.7$ Myr. Its decay produces a monoenergetic γ ray of energy $\epsilon_\gamma = 1.809$ MeV. Below is an all-sky map at 1.809 MeV, as seen by the COMPTEL instrument on the *Compton Gammma-Ray Observatory* space mission (Plüschke et al 2001; `arXiv:astro-ph/0104047`).



   (a) The COMPTEL map gives the γ-ray intensity or surface brightness I, i.e., the *number* of photons per unit area per unit time and per unit *angular area on the sky*; this angular area is given in steradians $\equiv$ sr = radian^2. To find the total γ-ray number flux F, roughly *estimate* the integral $F = \int I d\Omega$ of the surface brightness over angular area. Express your results in photons cm^{-2} s^{-1}.

   (b) Although the total ^{26}Al flux F comes from all across the Galactic disk, assume for simplicity it is all produced at the Galactic center, with a distance $d \simeq 10$ kpc. Assuming isotropic emission, show that this flux is proportional to the present number $\mathcal{N}$ of atoms and mass M of ^{26}Al in the Galaxy. Solve for the ^{26}Al mass, and express your answer in units of $M_\odot$. *Hint*: the γ-ray production rate is also the total ^{26}Al decay rate $\lambda\mathcal{N}$, where $\lambda = 1/\tau_{26}$ is the β-decay rate of ^{26}Al.

(c) Consider a model in which ^{26}Al is produced (at a constant rate) by supernovae throughout the galaxy. Let the rate of supernova explosion be $\mathcal{R}$, and the average mass of ^{26}Al produced and ejected by a supernova be $M_{\rm ej}$. Then show that the ^{26}Al mass M in the Galaxy changes with time according to

$$\dot{M} \equiv \frac{dM}{dt} = -\lambda M + \mathcal{R} M_{\rm ej} \tag{1}$$

(d) Solve eq. (1) exactly, and show that, regardless of the value of the initial ^{26}Al mass $M_0 = M(t=0)$, after times short compared to the Galaxy's lifetime, the ^{26}Al mass goes to a constant value

$$M \to M_{\rm eq} \equiv \frac{\mathcal{R}}{\lambda} M_{\rm ej} \tag{2}$$

which does not depend on M_0. Show that this value is what you get if the ^{26}Al mass is in a steady-state *equilibrium*, i.e., if $\dot{M} = 0$.

Hint: you may wish to consider the behavior of the *deviation* $\delta(t) = M(t) - M_{\rm eq}$.

(e) Theoretical models of supernova nucleosynthesis predict that Type II supernovae of different masses produce amounts of ^{26}Al with masses that span a large range $M_{\rm ej} \sim (10^{-5} - 10^{-4})M_\odot$. Use this, along with your measured value for M, to estimate $\mathcal{R}$, the Galactic supernova rate. Express your answer in events/century. Note that your result really measures the average Galactic supernova rate over some timescale; what is this timescale?

Other methods of estimating the present-day supernova rate typically give $\mathcal{R} \sim 3$ events/century. Compare your result to this and comment.

(f) Finally, one can also turn the problem around. Take $\mathcal{R} \sim 3$ events/century, find $M_{\rm ej}$, and comment on the implications for theoretical supernova nucleosynthesis models.

4. *A Cubic Shell Model*. The strong interaction between nucleons is quite complex, leading to the richness and challenge of nuclear theory. The crudest imaginable approximation to the nuclear interaction is as follows. Consider a "square-well potential" in the form of 3-D cubic box, with side length L: the potential is zero inside the box, and infinite outside of it.

(a) Solve the 3-D Schrödinger's equation to find the eigenstates for this potential, and use this to solve for the energies. You should find that there are three quantum numbers that describe the states, as befits a 3-D system. *Hint:* the result will bear a strong family resemblance to the 1-D "particle-in-a-box" eigenstates familiar from elementary quantum mechanics.

(b) If the particle is a neutron or a proton, find and sketch the 40 lowest energy states and their occupancy, remembering that the Pauli principle holds. Do you find any shell structure? If so, what are the magic numbers? What would be the lightest doubly-magic nucleus? How do the "cubic" magic numbers compare to the true magic numbers for real nuclei?