Astronomy 596/496 NPA Fall 2009
Problem Set #2
Due in class: Friday, Sept. 25

1. *Quantum Tunneling and the Coulomb Barrier.*

   (a) We first consider barrier penetration in one dimension. For the potential

   $$V(x) = \begin{cases} 0 & x < 0 \\ V_0 = const > 0 & x > 0 \end{cases} \tag{1}$$

   Solve the Schrödinger equation for a particle of mass m in this potential and having energy $E < V_0$. Show that for $x > 0$, the wavefunction is $\psi(x) = \psi(0)e^{-kx}$ with the wavenumber $k = \sqrt{2m(V_0 - E)}/\hbar$. Thus conclude that in this model the probability to penetrate a distance x in this potential is $P(x > 0) = P(0)\ e^{-2kx}$.

   (b) Now we generalize to the 2-body Coulomb problem. Find $R_{\rm cl}$, the classical radius of closest approach ("turnaround radius") given the particle's energy E and charge Z, for the case of zero angular momentum (i.e., for purely radial motion).

   (c) In a complete quantum calculation (which we won't do) the radial portion of wavefunction is similar to the 1-D case, but now $V(r) \neq constant$. Thus we should expect a wavenumber $k = k(r)$. One can show (by solving the Coulomb wavefunction or using the WKB approximation), that $|\psi(r)|^2 = |\psi(\infty)|^2 \exp(-2\pi\eta)$ where the exponential is the generalization of the previous expression

   $$2\pi\eta = 2 \int_R^{R_{\rm cl}} dr\ k(r) \tag{2}$$

   In solving the integral, you may find it useful to note that

   $$\int dx \sqrt{\frac{a}{x} - b} = \frac{a}{2\sqrt{b}} \left(\sin^{-1} u + \sqrt{1 - u^2} \right) \tag{3}$$

   where $u = (2bx/a) - 1$. If you are skeptical, or you are an integration jock, please feel free to verify this or to ignore this and do the integral your own way.

   We will be interested in the case in which the radius R at which we need to tunnel (i.e., the radius of the nuclear force, which is essentially the nuclear radius) is far inside the classical turnaround radius $R_{\rm cl}$. That is, we want the case in which $R \ll R_{\rm cl}$. Show that this condition is equivalent to the statement that the Coulomb barrier $E_C = Z_1 Z_2 e^2/R$ at the nucleus is much larger than the particle energy center-of-mass energy E.

   For this limiting case, show that

   $$2\pi\eta \approx \frac{2\pi Z_1 Z_2 e^2}{\hbar v} \tag{4}$$

   This is usefully written in terms of energy as $P(E) = e^{-bE^{-1/2}}$, as we saw in class. Find b in terms of Z_1, Z_2, A_1, A_2, and physical constants.

2. *Thermonuclear Lifetimes: Lithium in the Sun.* ^{7}Li in the Sun can be destroyed, mainly via the $^7\mathrm{Li}(p,\alpha)^4\mathrm{He}$ reaction. The thermonuclear rates for these reaction can be found in compilations on the web; for this problem I recommend the NACRE (2000) compilation which appears on the course `links` page. Note that rates are tabulated as $N_{\rm Avo}\langle\sigma v\rangle$, in units cm^3 s^{-1} g^{-1}, and $T_9 = T/10^9$ K.

   (a) Estimate the temperature $T_{\rm burn}$ at which ^{7}Li is destroyed in the Sun–i.e., find T such that the mean life for ^{7}Li against destruction is $\leq t_\odot \simeq 5$ Gyr. Take an average solar density of $\bar{\rho} = 1.4$ g cm^{-3}, and hydrogen mass fraction $X_\odot \sim 0.7$. How does $T_{\rm burn}$ compare to the central temperature of the Sun, $T_{\rm c,\odot} = 16 \times 10^6$ K? *Hint:* You may find that the tabulated rates are too sparse to get a good estimate, in which case you should use the analytic expression given. Don't be afraid to approximate: while the analytical rate expression is complicated, note that for temperatures $T_9 \ll 1$, it simplifies something manageable.

   (b) From your handy Anders & Grevesse abundance chart, find the difference between photospheric and meteoritic Li abundance. In light of the results above, discuss possible implications of the discrepancy for the nature of the Sun.

3. *Conservation Laws.* Consider the following processes. For each reaction, state whether the reaction is allowed or forbidden, according to the conservation laws we have discussed. If the reaction is forbidden, indicate which law(s) are violated. If the reaction is allowed, indicate if it is endothermic or exothermic. (Refer to the Particle Data Group webpage for particle masses; you may take $m_\nu = 0$.)

   (a) $p + \gamma \rightarrow p + \pi^0$

   (b) $\bar{\nu}_\tau + \tau^- \rightarrow \mu^- + \bar{\nu}_\mu$

   (c) $\pi^- + p \rightarrow n + \pi^0$

   (d) $\pi^+ + n \rightarrow p$

4. *The Friedmann Equation.* The Friedmann equation governs $\dot{a}$, while the Friedmann acceleration equation governs $\ddot{a}$. Take the time derivative of the Friedmann equation and show that the Friedmann acceleration equation (with pressure!) holds if we have $d(\varepsilon a^3) = -p\, d(a^3)$, where $\varepsilon = \rho c^2$ is the cosmic energy density; i.e., we see that energy conservation or the First Law of Thermodynamics holds.

5. *A Matter-Dominated Universe.* The simplest and most intuitive cosmological model is for a flat universe whose density is dominated by that of non-relativistic matter. Hwere we will study this important case, known as the Einstein–de Sitter universe or matter-dominated universe, in detail.

   (a) Consider a matter-dominated universe, with present mass density ρ_0. For such a universe, use (but do not solve!) the Friedmann equation to find the present value H_0 of the Hubble parameter in terms of ρ_0 and physical constant(s).

   (b) Now solve the Friedmann equation to find $a(t)$, subject to the boundary conditions $a(0) = 0$ and $a(t_0) = 1$. In doing this, you should find a relationship between t_0 and H_0, and thus between t_0 and ρ_0.

(c) Sketch or plot $a(t)$. Physically interpret this behavior in terms of the history and fate of the universe.

Find the cosmic acceleration $\ddot{a}/a$. Interpret your result (and it's sign!) physically.

Finally, find $\Omega(t)$ in this model, and interpret your result physically.

(d) Find $H(t)$ and $z(t)$ for a matter-dominated universe. Is the Hubble "constant" aptly named in this model? If not, describe and physically interpret the evolution of $H(t)$. What is the age of a matter-dominated universe at $z = 1$, expressed in terms of t_0? What is the redshift at which such a universe is at 10% of its present age?

(e) Find $\rho(t)$, and interpret the behavior physically. Show that you can express your answer entirely in terms of t and physical constant(s).

(f) Using the current best measured value of H_0, find t_0 for a matter dominated universe. Compare your answer to the age of the Earth, and also to the age of the oldest stars (gobular clusters): $t_{\rm gc} \geq 12$ Gyr. Comment on the implications of this comparison.