Astronomy 596/496 NPA Fall 2009
Problem Set #3
Due: Friday, October 9

1. *Particle collision energetics.*

   (a) At CERN (located outside of Geneva, Switzerland: see links on course page), an experiment predating the LHC accelerated to high energies counter-rotating beams of e^- and e^+ (equal energies for the two species) which were brought into head-on collisions. This is known as a "colliding beam" experiment. If these particles have sufficient energy, they can annihilate to create muon-antimuon pairs: $e^+e^- \to \mu^+\mu^-$. Find an expression for the minimum *kinetic* energy each e must have (threshold) in order for this reaction to occur. (The muon mass and other properties can be found at the Particle Data Group site linked off the course page.)

   (b) Now consider a "fixed target" experiment, in which an energetic beam of e^+ collides with matter (which of course includes e^-) at rest. Now find the *kinetic* energy threshold for $e^+e^- \to \mu^+\mu^-$. Why is this answer different from that of part (a)?

2. *Statistical Mechanics.*

   (a) Consider a non-relativistic particle, the "nucleon," with mass m_u. If the baryon number density is n_B, find an expression for the chemical potential μ_B in terms of n_B, T, and m_u.

   (b) For a relativistic species ($m, \mu \ll T$), show that

   $$\rho_{\rm fermion} = \frac{7}{8}\,\rho_{\rm boson} \tag{1}$$

   for particles with the same g. Hint: it is probably simplest to just massage the expression for $\rho_{\rm fermion}$ into a multiple of the expression for $\rho_{\rm boson}$. The ways I know to do this involve judicious rewriting of the integrand; one of these involves recalling that $1/(1-x) = \sum_{n=0}^{\infty} x^n$ for $x < 1$. The other way involves manipulation of the difference $f_{\rm b} - f_{\rm f}$ between the Bose-Einstein and Fermi-Dirac distributions.

   (c) Bonus: also show that $n_{\rm fermion} = 3/4\; n_{\rm boson}$.

3. *Measures of Cosmic Baryons*

   (a) Derive and compute the conversions among the baryon-to-photon ratio η and the present values of the baryon density $\rho_{\rm B,0}$ as well as the baryon density parameter $\Omega_{\rm B,0}h^2$. You may use the present photon temperature $T_0 = 2.725$ K (and the associated photon number density). You may also assume nuclear binding energies $B_i \ll m_u$.

(b) Show that the primordial ^{4}He mass fraction Y_p can, to an excellent approximation, be written as

$$Y_p = \frac{2(n/p)}{1+(n/p)} \tag{2}$$

Where (n/p) is the neutron-to-proton ratio right before BBN light element production. What assumption(s) go into this result?

4. *Baryons, Photons, and Entropy.*

(a) The usual first law of thermodynamics for a closed system of one type of particles is $TdS = dE + pdV$. If there are reactions which can change the number of particles, then this is modified to become

$$TdS = dE + pdV - \mu dN \tag{3}$$

where N is the number of particles in volume V, and μ is the chemical potential. If there are multiple species, then this expression holds for the entropy of each species, and the total entropy is the sum of all of the entropies of each species. Use eq. (3) to derive an expression for the entropy density $s = (\partial S/\partial V)_T$ in terms of the (energy) density $\varepsilon = \rho c^2 = (\partial E/\partial V)_T$ and p.

(b) Apply the result of (a) to our universe, with $\eta \ll 1$, and $\mu/T \ll 1$ for relativistic species, but μ not necessarily negligible for the baryonic species. Show that $s_{\rm tot} \simeq s_{\rm rel}$ (i.e., $s_{\rm non-rel} \ll s_{\rm rel}$ always). This in means that the in the universe is locked into CMB photons. Go on to show that

$$s_{\rm rel} = g_{*,S}\frac{2\pi^2}{45}k\left(\frac{kT}{\hbar c}\right)^3 \tag{4}$$

where $T = T_\gamma$, and

$$g_{*,S} = \sum_{\rm bosons} g_i (T_i/T)^3 + \frac{7}{8}\sum_{\rm fermions} g_i (T_i/T)^3 \tag{5}$$

(c) Pair annihilation $e^+e^- \to \gamma\gamma$ occurs when $T \sim m_e \sim 0.5$ MeV, and increases the number of cosmic photons n_γ. This is after weak freezeout, so the photons are "heated" but not the neutrinos. From the fact that the comoving neutrino entropy $S_\nu = s_\nu a^3$ is conserved, show that $T_\nu \propto a^{-1}$ exactly. Then equating the EM (photon + pair) comoving entropy before and after pair annihilation, show that $T_{\gamma,f}/T_{\gamma,i} = (11/4)^{1/3}\ a_i/a_f$; use this to show that today,

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma = 0.714 T_\gamma = 1.95\ \mathrm{K} \tag{6}$$

Thus, conclude that today,

$$s \simeq 7.0\ k\ n_\gamma \tag{7}$$

Evaluate the dimensionless *entropy* per baryon $s/kn_{\rm B}$ today, and comment.

5. *Weak reaction cross sections.*

Fermi's Golden Rule says that the rate (probability per unit time) λ for a quantum system to make a transition from initial state i to final state f is

$$\lambda = \frac{2\pi}{\hbar}|M_{fi}|^2\frac{dN_f}{dE} \tag{8}$$

where the "matrix element" $M_{fi} = \langle f|H_{\rm int}|i\rangle$ is the expectation value of the interaction potential, and dN_f/dE is the number of final states per unit energy.

(a) We wish to apply this formula to the case of the weak conversion $\nu_e + n \rightarrow p + e$. We thus need to know how to compute the matrix element. In general, this is the overlap between the initial and final wavefunctions when acted on by the perturbing interaction potential $H_{\rm int}$:

$$M_{fi} = \langle f|H_{\rm int}|i\rangle = \int d^3r_1\ d^3r_2\ \Psi_f^*(\vec{r}_1,\vec{r}_2)\ H_{\rm int}(\vec{r}_1,\vec{r}_2)\ \Psi_i(\vec{r}_1,\vec{r}_2) \tag{9}$$

where the initial state wavefunction is the product of the individual particle wavefunctions (not necessarily at the same position): $\Psi_i(\vec{r}_1,\vec{r}_2) = \psi_n(\vec{r}_1)\ \psi_\nu(\vec{r}_2)$ and the final state wavefunction is $\Psi_f(\vec{r}_1,\vec{r}_2) = \psi_p(\vec{r}_1)\ \psi_e(\vec{r}_2)$.

For the weak interaction at low energies, the interaction potential between the nucleon and the lepton is so short range that it can be approximated as

$$H_{\rm int}(\vec{r}_1,\vec{r}_2) = 2G_{\rm F}\ \delta^3(\vec{r}_1 - \vec{r}_2) \tag{10}$$

where $\delta^3(\vec{x})$ is the 3-dimensional Dirac δ function and $G_{\rm F}$ is a constant of nature, the "Fermi coupling constant," found in the Particle Data Group (follows "Reviews, Tables, ..." link. then "Constants" link).[1] Physically, this form of the potential corresponds to the assumption that the interaction has *zero* range–i.e., that the particles have to touch to interact–thus, this potential is know as a "contact potential."

Use this potential to find the matrix element for a pair of interacting particles within a box of volume V. You should use the usual free particle wavefunctions

$$\psi(\vec{r}) = \frac{1}{\sqrt{V}}e^{i\vec{p}\cdot\vec{r}} \tag{11}$$

with "box normalization" as indicated.

(b) We will adopt the (very good) approximation in which the n and p are at rest. Now the initial state is one with a fixed input ν energy E_ν, and the number of final states is $dN_e = 4\pi V p_e^2 dp_e/(2\pi\hbar)^3 = \rho(E_e)dE_e$. In this approximation, find λ_ν.

(c) Use the definition of cross section (in terms of λ_ν) to infer $\sigma(E_\nu)$. Note that, in the 2-body process we consider, the initial neutrino density is $n_\nu = 1/V$. If all has gone well, the unsightly factors of box volume V will disappear. You may assume $E_e, E_\nu \gg m_e c^2, Q = m_n - m_p$.

(d) If we write $\sigma(E_\nu) = \sigma_0(E_\nu/m_e)^2$, find an expression for σ_0, and compute its numerical value, comparing to the one given in class.

[1] The factor of 2 in the right side of eq. (10) is a historical accident, the explanation of which I would be happy to discuss over a beverage.