Astronomy 596/496 NPA Fall 2009
Problem Set #4

Due: Friday, October 23

1. *Probing Physics with BBN: Variations in Fundamental Constants.* Late in his career, Dirac postulated that Newton's gravitational constant $G_{\rm N}$ may evolve with cosmic time. This idea has recently received a great deal of attention since (1) some quantum gravity theories (including versions of string theory) predict evolution in fundamental "constants," and (2) observations of metal lines in quasar absorption systems seem to suggest that at $z \sim 3$, the fine structure constant α differs from its current value by a tiny but nonzero amount: $\delta\alpha_{z=3}/\alpha_{z=0} \sim -1 \times 10^{-5}$.

   Consider the case of changing $G_{\rm N}$ (but no change in the fine-structure constant or anything else). Namely, imagine that at the time of BBN, $G_{\rm N}$ were different from its present value G_0, by some amount δG; i.e., at BBN, we have $G = G_0 + \delta G$. Much as in the "neutrino counting" argument discussed in class, we can use light elements to probe δG in the early universe.

   (a) In the neutrino counting discussion in class (and in the Director's Cut Extras) we saw how adding $\delta N_\nu = N_\nu - 3$ extra neutrio species leads to an *increase* in the primordial ^{4}He abundance (mass fraction) δY_p. Now let's fix $N_\nu = 3$ as in Standard BBN, but now allow for $\delta G \neq 0$. If during BBN we had $\delta G > 0$, would this lead to an *increase* or *decrease* in the primordial ^{4}He abundance? Explain your reasoning.

   (b) As discussed in class, in case where $\Delta N_\nu \neq 0$ and $\delta G = 0$, the effect of adding neutrinos is to change the expansion rate, which ultimately affects Y_p. Now look at our case in which $\Delta N_\nu = 0$ and $\delta G \neq 0$, and explain why here again, the resulting effect is to change the expansion rate. Go on to explain why we can relate the two cases (extra neutrinos vs modified gravity strength) via

   $$\frac{7}{4}\left(\frac{\delta N_\nu}{g_*^{\rm std}}\right)_{G=G_0} = \left(\frac{\delta G}{G_0}\right)_{N_\nu = N_\nu^{\rm std}=3} \tag{1}$$

   where $g_*^{\rm std}$ is the pre-freezeout effective number of relativistic degrees of freedom in Standard BBN, and G_0 is standard laboratory value today. *Hint*: compare the perturbation $\delta H/H$ to the expansion rates in the two cases.

   (c) As shown by, e.g., Cyburt, Fields, Olive, & Skillman (2005), the WMAP measurement of η can be combined with BBN theory and observations of deuterium to constrain N_ν (in the case of $\delta G = 0$). Use this limit, $N_\nu < 4.44$ at the 95% CL, and the result from part (b) to deduce a bound on $\delta G/G_0$ at the time of BBN (in the case of $N_\nu = 3$). What is this limit? Comment on your result.

2. *The Planck Mass.*

   (a) In (special) relativity, a particle of mass m has a characteristic energy scale, mc^2, associated with it, and an characteristic momentum scale mc. In quantum mechanics there is a natural length scale associated with a particle of momentum p, namely the de Broglie wavelength $\lambda \sim \hbar/p$. One thus arrives at a lengthscale associated with relativistic quantum effects, namely the Compton wavelength $\lambda_c = \hbar/mc$. At length scales at or smaller than this, one expects relativistic quantum effects to be important, and in fact this is how we calculated the range of forces mediated by massive bosons (e.g., the weak force).

In (classical) General Relativity, there is a natural length scale associated with a body of mass m, namely $r_{\rm gr} \sim Gm/c^2$ (this is half of the so-called Schwarzschild radius) . For the known fundamental particles, it turns out that $r_{\rm Sch} \ll \lambda_c$, which means that one may ignore General Relativity when describing them. However, if a particle had a mass m such that $r_{\rm Sch} = \lambda_c$, one would need a full General Relativistic quantum theory to describe the particle–i.e., *quantum gravity*. This mass scale is known as the Planck mass $M_{\rm Pl}$. Calculate $M_{\rm Pl}$ and give its value in energy units of GeV. Also calculate the associated Planck time and Planck length. These correspond to the temperature, age, and size of the Universe before which quantum gravity effects must be included, and thus mark the extreme edge of applicability of current (non-quantum-gravity) theories.

(b) Verify that in natural units, for a radiation-dominated universe, we have an expansion rate $H \sim T^2/M_{\rm Pl}$.

3. *Hot Relics: Neutrinos.* Assume that the known neutrino species (e, μ, τ) have masses such that $m \ll 1$ MeV, but $m \gg T_0$.

(a) In class we considered cold relics, for which $T_f \ll m$. Neutrinos, however, are hot relics. If species ψ has $T_f \gg m$ and $\mu_\psi = 0$, find the thermodynamic equilibrium abundance $Y_{\rm eq}(x)$.

(b) For all species of neutrinos, the annihilation cross section is of the same order as the $n \leftrightarrow p$ cross section you derived in the last problem set: $\sigma_{\rm ann} \simeq \sigma_0 (E/m_e)^2$. Use this to estimate an expression for $\langle \sigma v \rangle_{\rm ann}$ as a function of T and of $x = m_\nu/T$.

(c) Using the result from (b), find the neutrino freezeout temperature x_f. If each species i has mass m_i, find its present relic abundance (assuming it is non-relativistic today, $m_\nu \gg T_0$).

(d) Use the result from (c) to show that the condition $\Omega_\nu \leq 1$ corresponds to a limit on neutrino mass,

$$\sum_{\rm neutrinos} m_i \lesssim 100 \text{ eV} \tag{2}$$

How does this compare to Particle Data Group constraints on neutrino masses?

4. *Cold Relics: A Gut Feeling for WIMP Abundances.* Compute the local mass density $\rho_0 \equiv \rho_{\rm w}(R_0)$ of WIMPs at the Sun's Galactocentric radius $R_0 \sim 10$ kcp Assume that the Galactic rotation curve (circular speed V vs R) is "flat" at and around our location, i.e., $V(R) = V_0 \simeq 220$ km/s. (Along the way, you will want to use Gauss' law to find the mass $M(R)$ inside R.) You should find $\rho_0 \simeq 0.3$ GeV cm^{-3}. If WIMPs have a mass $m_{\rm w} \sim 100$ GeV, estimate the average number of WIMPs in your body at any given time.

5. *Cold Relics: WIMP Detection.* In class we saw that particles with annihilation cross sections $\sigma_{\rm ann} \sim 10^{-36}$ cm^2 are attractive candidates for non-baryonic dark matter. If these WIMPs are indeed the dark matter, there is much knowledge and profit to be gained by detecting them. In this problem, we will assume that the Galactic dark matter is in the form of WIMPs, and examine two methods for discovering them.

   Please answer *at least one* of these two; if you answer the other, it will be a *bonus* point.

   (a) The center of our Galaxy is where the dark matter density should be the highest. The nature of the dark matter density profile at the Galactic center is highly controversial, but for simplicity we will use the famous Navarro-Frenk-White (NFW; 1997 ApJ 490, 493) profile:

   $$\rho(r) = \rho_0 \frac{\beta}{r/R_0(1+\alpha r/R_0)^2} \tag{3}$$

   where $\alpha = 3.7$, and $\beta = (1+\alpha)^2$. Note that while the central density is formally divergent, the enclosed mass $M(r)$ is finite as $r \to 0$. We will be interested in the inner Galaxy ($r \lesssim R_0/\alpha$), where we can approximate $\rho(r) \simeq \beta\rho_0 R_0/r$.
   The WIMP annihilation rate per unit volume is $q_{\rm ann} = \sigma_{\rm ann} v n_{\rm w}^2$. Estimate the WIMP speed v using the circular speed $V(r)$, and use this to calculate the total annihilation rate $\dot{N}_{\rm ann} = \int q(r) d^3r$ for the inner Galaxy.
   Then assume the annihilations go to photons (γ-rays). Taking the inner Galaxy to be a point source to γ-rays, estimate the annihilation photon flux (by number) Φ_γ at Earth. The recently-launched *Fermi Gamma-Ray Space Telescope* measures γ rays with energies ~ 30 MeV to ~ 300 GeV, and has point source sensitivity of $\Phi_{\rm min,Fermi} \simeq 10^{-9}$ cm^{-2} s^{-1}. Do you expect WIMP annihilations to be observable by *Fermi*? What could we learn about the properties of WIMPs if an annihilation signal *were* observed by *Fermi*?

   (b) Terrestrial WIMP experiments are sensitive to WIMP-nucleon *scattering* events, rather than annihilations. These occur due to the WIMP flux $\Phi_{\rm w} \simeq n_{\rm w,0} V_0$ through the Earth. Given a detector of mass $M_{\rm det}$ made of nuclei of mass number A, and a WIMP-nucleus scattering cross section σ_{W-A}, show that the total scattering event rate $\mathcal{A}$ in the detector scales linearly with the detector mass: $\mathcal{A} \propto M_{\rm det}$. Thus a detector-independent quantity[1] is the rate per unit mass, $R = \mathcal{A}/M_{\rm det}$. Find an expression for R, and evaluate it for the case when $\sigma_{W-A} = \sigma_{\rm ann}$, for a WIMP mass $m_{\rm w} \sim 100$ GeV and for an iodine detector with $A \sim 100$. Express your results in units of [events day^{-1} kg^{-1}].
   Current detector capabilities can probe event rates of about 0.03events day^{-1} kg^{-1}. Comment on the ability of these detectors to detect a WIMP signal. In fact, no signal is seen; what does this imply for the WIMP interaction strength σ_{W-A}?

[1] In fact, this is independent of the detector *mass*, but can and does depend on the detector composition (A), since it is expected that some nuclei are more favorable for WIMP detection than others.