Astronomy 596/496 NPA Fall 2009
Problem Set #5
Due: Friday, Nov. 6

1. *Neutrino Oscillations: Two-Flavor Approximation.* Here we will get a sense of the workings of neutrino flavor change by looking at a system where there are two neutrino flavor states (call them $|\nu_e\rangle$ and $|\nu_x\rangle$), which are weak eigenstates (i.e., eigenstates of the weak Hamiltonian). What this means is that

$$\langle \nu_e|\nu_e\rangle = \langle \nu_x|\nu_x\rangle = 1 \text{ and } \langle \nu_e|\nu_x\rangle = \langle \nu_x|\nu_e\rangle = 0 \tag{1}$$

On the other hand, there are two mass states $|\nu_1\rangle$ and $|\nu_2\rangle$, which are vacuum eigenstates (i.e., eigenstates of the vacuum Hamiltonian), so that

$$\langle \nu_1|\nu_1\rangle = \langle \nu_2|\nu_2\rangle = 1 \text{ and } \langle \nu_1|\nu_2\rangle = \langle \nu_2|\nu_1\rangle = 0 \tag{2}$$

The weak eigenstates are not mass eigenstates. Thus it is not meaningful to speak of the masses of the flavor states, only of the mass states.

A general neutrino quantum state can be expressed as a superposition of the two flavor states, *or* of the two mass states. That is, the pair of flavor states and the pair of mass states each can serve as a *basis* for the generic neutrino state. This is similar to the case of a spin-1/2 particle, where a general spin state can be expressed as a superposition of "up" and "down" states or of "left" and "right" states.

(a) We wish to understand the relationship between flavor and mass states. According to the rules above, we may generally write, for any instant of time,

$$|\nu_e\rangle = a|\nu_1\rangle + b|\nu_2\rangle \tag{3}$$
$$|\nu_x\rangle = c|\nu_1\rangle + d|\nu_2\rangle \tag{4}$$

where $u = a, b, c, d$ are time-independent numbers that describe the "mixing" of mass and flavor states; we may take a, b, c, d to be real. If flavor states *were* mass eigenstates, what would be the values of a, b, c, d? In light of this, what is the physical significance of each of the (universal) dimensionless constants a, b, c, d? Also show that the normalization conditions for $|\nu_e\rangle$ and $|\nu_x\rangle$ give $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$.

(b) Now rewrite eqs. (3) and (4) to solve for $|\nu_1\rangle$ and $|\nu_2\rangle$. You should find

$$|\nu_1\rangle = \frac{d}{ad-bc}|\nu_e\rangle - \frac{b}{ad-bc}|\nu_x\rangle \tag{5}$$
$$|\nu_2\rangle = -\frac{c}{ad-bc}|\nu_e\rangle + \frac{a}{ad-bc}|\nu_x\rangle \tag{6}$$

Then show that the normalization conditions for $|\nu_1\rangle$ and $|\nu_2\rangle$ give $ad - bc = 1$, as well as $a^2 + c^2 = 1$ and $b^2 + d^2 = 1$. Consequently we find $a = d$ and $b = -c$.

(c) In light of part (b), show that it is possible to write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{7}$$

What is the connection between θ, a, and b? The angle θ is not one in physical space, but what is its physical significance?

(d) Using eq. (7), and following class notes, show that the general time dependence of the flavor states is

$$|\nu_e\rangle_t = \cos\theta e^{-iE_1 t/\hbar}|\nu_1\rangle_{t=0} + \sin\theta e^{-iE_2 t/\hbar}|\nu_2\rangle_{t=0} \tag{8}$$

and similarly for $|\nu_x\rangle$.

(e) Now find the probability $P_{\nu_e \to \nu_e}(L)$ that a neutrino born as a ν_e in a weak interaction at the origin is detected as a ν_e after traveling in vacuum over a distance $L \approx ct$. Verify that the result is the oscillatory expression given in class.

(f) All of this hifalutin theory now needs to be compared with experiment. In Japan, KamLAND (see links from lecture pages) measures reactor (anti)neutrinos which travel a mean disance $L \sim 180$ km. Their most recent results are summarized in Abe et al.: KamLAND Collaboration (2008) Phys. Rev. Lett., 100, 221803; `arXiv:0801.4589`). From Figure 1 of Abe et al., make an estimate of their measured value of $P_{\nu_e \to \nu_e}$ at $E_\nu = 3$ MeV and $E_\nu = 6$ MeV. Using these (and more data points if you wish), estimate Δm_{12}^2 and $\sin^2 2\theta$, noting that these are the same for antineutrinos as for their neutrino counterparts.

How does your estimate compare to their advertized result? How does your estimate compare with the parameters needed to solve the solar neutrino problem?

2. *SN 1987A: Light Curve and Nucleosynthesis.* Bouchet et al. (1991 A&A 245, 490) sum the UV, optical, and IR light curves for SN1987A to arrive at a bolometric light curve. These appear in their Table 6 and Figure 3, where we see that in late times (i.e., after about 140 days), the supernova dimmed in a manner that is roughly linear in the semi-log plot. This trend encodes important information about iron nucleosynthesis in SN 1987A.

(a) Show that the late-time light curve can be well-fit to the form $L(t) = L(0)\ e^{-t/\tau}$. Use the data to estimate the $L(0)$ and the time constant τ. Remember that the luminosity is given in the form of base 10 logs. Both of your fit parameters have a story to tell.

(b) At late times, it is expected that the energy source powering the supernova comes from the decay of ^{56}Ni. This decay takes two steps, first $^{56}\text{Ni} \to {}^{56}\text{Co}$, then $^{56}\text{Co} \to {}^{56}\text{Fe}$. Each step involves a β-decay, usually into an excited state of the daughter nucleus which then γ-decays to the ground state. Show that at late times, we need only consider the ^{56}Co decay. Then compare the observed time constant τ from part (a) with the ^{56}Co mean life τ_{56} (not half-life!). Comment on the results.

(c) Assuming that at late times, the supernova is powered *entirely* by ^{56}Co decay, use $L(0)$ from part (a) to infer the ^{56}Ni (and ultimately ^{56}Fe) yield of SN 1987A. Note that the energy released per decay depends on the details of how the β-decay proceeds (electron capture versus e^+ emission) and into which excited $^{56}\mathrm{Fe}^*$ state the decay goes. To make a long story short, it turns out that the mean energy release per decay (not in the form of neutrinos) is about 3.8 MeV. Using this, you should find a ^{56}Fe yield close to the now-canonical result, $m_{\mathrm{ej,Fe}} \sim 0.07 M_\odot$.

(d) *Bonus*: You may notice that at times after about 300 days, the light curve is not perfectly exponential, but falls too rapidly. In other words, despite the presence of decaying ^{56}Co, not all of its energy is powering the light curve. What might be going on?