Announcements:
• Problem Set 1 due next time
  Note: intensity units in figure are $10^{-3}$ photons cm$^{-2}$ s$^{-1}$ sr$^{-1}$
  for example: red $\rightarrow \sim 1.2 \times 10^{-3}$ photons cm$^{-2}$ s$^{-1}$ sr$^{-1}$
• TA Office Hours: 3-4pm Thursdays
  ...or by appointment

Last time: nuclear reactions and cross sections

Q: how is a cross section defined?
Q: what is the physical significance of $\sigma$? units?
Q: how are cross sections related to particle sizes?
Cross Sections: Post-Holiday Reminder

For reaction $a + b \rightarrow \text{something}$ cross section defined as

$$\sigma_{ab} \equiv \frac{\text{reaction rate per target}}{\text{projectile flux}} = \frac{\Gamma_{\text{per } b}}{j_a} = \frac{\Gamma_{\text{per } a}}{j_b}$$

(1)

where the flux of species $i$ with speed $v$ is

$$j_i = n_i v = \# \text{ particles area}^{-1} \text{ time}^{-1}$$

(2)

Physically: cross section $\sigma_{ab}$ is projected size of “bullseye” each reactant “sees” in the other

⇒ measures reaction probability/strength

Cross section controlled by physics of $a + b$ interaction

- if short-ranged “contact” interaction: $\sigma \sim (\text{particle size})^2$
- if long-ranged interaction: $\sigma$ unrelated to particle size

Q: $\sigma$ when distribution of speeds? most important example?
Reaction Rates for Thermal Particles

for $ab \rightarrow cd$, reaction rate per volume: $r_{ab \rightarrow cd} = n_a n_b \langle \sigma_{ab} v \rangle$

where $\langle \cdots \rangle$ is average over distribution of relative $v$

For non-relativistic gas at temperature $T$:
relative speeds are in *Maxwell-Boltzmann* distribution

$$\langle \sigma v \rangle = \sqrt{\frac{8}{\pi \mu}} \left( \frac{1}{kT} \right)^{3/2} \int_0^\infty dE \ E \ \sigma(E) \ e^{-E/kT}$$

(3)

Note $e^{-E/kT}$: exponentially suppresses energies $E \gtrsim kT$

Recall Coulomb barrier:

$$E_C = Z_1 Z_2 e^2 / r = Z_1 Z_2 \ 1.44 \text{ MeV} \ (1 \text{ fm} / r)$$

Classically, need $E \gtrsim 1$ MeV to overcome barrier
But $kT = 0.86 \text{ keV} \ (T/10^7 \text{ K}) \ll E_C$ for solar-like temp !?!

Q: What does this seem to imply?
Q: What's the flaw?
Quantum Mechanics to the Rescue

nuclei are quantum particles → can tunnel
Probability to tunnel under Coulomb barrier:

\[ P \propto e^{-2\pi Z_1 Z_2 e^2 / \hbar v} = e^{-bE^{-1/2}} \]  

(4)

Also: geometrical factor: cross section \( \sigma \propto \lambda_{\text{deB}}^2 \), \( \lambda_{\text{deB}} = \hbar / p \) de Broglie wavelength
\[ \Rightarrow \sigma \propto 1/p^2 \propto 1/E \]

expect \( \sigma \) functional form

\[ \sigma(E) = \frac{S(E)}{E} e^{-2\pi \eta} = \frac{S(E)}{E} e^{-bE^{-1/2}} \]

\( S(E) \): “astrophysical \( S \)-factor”

• \( S(E) \) encodes nuclear contribution to reaction
• \( S(E) \) often slowly varying with \( E \)
  
  Q: if so, \( \sigma \) behavior at large \( E \)? small \( E \)?
• \( \sigma \) and S-factor for \( ^3\text{He}(\alpha, \gamma)^7\text{Be} \)
Thermonuclear Rates

So: thermonuclear rates reduce to:

\[ \langle \sigma v \rangle = \langle \sigma v \rangle_T = \frac{8}{\pi \mu (kT)^{3/2}} \int_0^\infty dE \ S(E) e^{-E/kT - bE^{-1/2}} \]

Procedure:
1. astro theory/obs identifies needed reaction
2. nuclear expt: measure \( \sigma(E) \rightarrow S(E) \)
3. find \( \langle \sigma v \rangle \) vs \( T \) (usually numerically)
4. fit result to function

Q: note exponential–behavior vs \( E \)? implications?
The Gamow Peak

The integrand $S(E)e^{-G(E)}$ peaks at/near minimum of exponential

$G(E) = E/kT + bE^{-1/2}$

min at $G' = 0$: “most effective energy” or “Gamow Peak”

$E_0 = (bkT/2)^{2/3}$, where

$$G_{\text{min}} \equiv \tau = G(E_0) = 3(b^2/4kT)^{1/3}$$

$$= 4.25(Z_1^2Z_2^2A)^{2/3} \left( \frac{10^9 \text{ K}}{T} \right)^{1/3}$$

Q: behavior with $T$? Interpretation?

Expand exponential around peak energy $E_0$:

$$G(E) \approx \tau + \frac{1}{2}G''(E_0)(E - E_0)^2$$
use expansion of exponential
\[ G(E) \approx \tau + \frac{1}{2} G''(E_0) (E - E_0)^2 \]
in thermonuclear integral (method of steepest descent)

Then we have

\[
\langle \sigma v \rangle \approx \sqrt{\frac{8}{\pi \mu (kT)^{3/2}}} S(E_0) e^{-\tau} \int_{-\infty}^{\infty} dE \ e^{-(E-E_0)^2/2\Delta^2}
\]

\[
= \frac{8}{9\sqrt{3\pi Z_1 Z_2 e^2 m}} \tau^2 e^{-\tau} S(E_0)
\]

\[
\propto T^{-2/3} e^{-a/T^{1/3}}
\]

Q: behavior at high \( T \)? low \( T \)? are these reasonable?
Mean Lifetimes

for reaction $i + j \rightarrow k + l$
define mean lifetime $\tau_i(ij)$ of $i$ against reaction with $j$ as

$$\dot{n}_i = -\frac{n_i}{\tau_i(ij)}$$

(8)

or $\tau_i(ij) = \|n_i/\dot{n}_i\|$

But $\dot{n}_i = -r_{ij} = -n_in_j\langle\sigma v\rangle_{ij}$

$\Rightarrow \tau_i(ij) = 1/n_j\langle\sigma v\rangle_{ij} = 1/\Gamma_{per i}(ij)$

useful to write

$$\Gamma_{per i}(ij) = n_j\langle\sigma v\rangle_{ij} \simeq \frac{X_j}{A_j} \frac{\rho}{m_u}\langle\sigma v\rangle_{ij} = \frac{X_j}{A_j}\rho[ij]$$

(9)

where $[ij] = \langle\sigma v\rangle_{ij}/m_u = N_{Avo}\langle\sigma v\rangle_{ij}$ given in tabulations

Why would this be a useful form?
Partial Lifetimes: Examples

Reactions in the Sun

In the solar core today:

\[ T \approx 16 \text{ MK} = 1.6 \times 10^7 \text{ K} \]
\[ \rho \approx 150 \text{ g cm}^{-3} \]
\[ X_H \approx 0.33 \]

What is lifetime of a deuteron against \( d(p, \gamma)^3\text{He} \)?

\[
\frac{1}{\tau_d(pd)} = X_H \rho [dp \rightarrow \gamma^3\text{He}]
\]
\[
\approx (0.3)(150 \text{ g cm}^{-3})(2 \times 10^{-10} \text{ cm}^3 \text{ s}^{-1} \text{ g}^{-1})
\]
\[
\approx 10^{-8} \text{ s}^{-1}
\]

or \( \tau_p(pd) \approx 3 \text{ yrs: “immediately”} \)
compare $^{16}\text{O}(p, \gamma)^{17}\text{F}$

Exponential factor $\tau(E_0) \sim 4 \times$ larger!

$\tau_{^{16}\text{O}}(p^{^{16}\text{O}}) \sim 10^{57}$ s $\gg$ age of Univ!

no $^{16}\text{O}$ burned in solar core (on main seq)
Particle Physics
Fundamental result of Relativistic QM: every particle has an antiparticle

- $\bar{e}^- = e^+$ positron
- $\bar{p} = \text{antiproton}$

Fermilab: $p\bar{p}$ collisions

antimatter is not second class citizen!

- e.g.: $e^+$ totally stable when left alone

So why so volatile in the lab?

www: $e^+$ annihilation in Galactic center