Astro 596/496 NPA
Lecture 11
Sept. 18, 2009

Announcements:

- Problem Set 2 due Friday
  Note on nuke reaction rate databases (all on course `links`):
  NACRE link now fixed; you are also welcome to use
  instead the more up-to-date Reaclib database at JINA
- Next semester, ASTR 596/496 PC: Physical Cosmology
  complementary to this course

Last time: *Al Friedmann's amazing equation*

Q: Friedmann (energy) eq is...?

Q: what factors are constant? what are variable?

Friedmann (Energy) Equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{\kappa c^2}{R^2 a^2} \tag{1}$$

variables change with time

a: cosmic scale factor

ρ: total cosmic mass-energy density

constants fixed for all time

$\kappa = \pm 1$ or 0: sign of "energy" (curvature) term

R: characteristic lengthscale, GR $\rightarrow$ curvature scale

Q: how does expansion of U depend on contents of U?

Q: how does expansion of U not *depend on contents of U?*

Q: what about acceleration–$\ddot{a}$?

Friedmann Acceleration Equation

Newtonian analysis gives $\ddot{a}$ for $P \to 0$

In full GR: with $P \neq 0$, get Friedmann *acceleration* eq.

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G(\rho + 3P/c^2) \tag{2}$$

Pressure and Friedmann

★ in "energy" $(\dot{a})$ eq.: P *absent*, even in full GR

★ in acceleration eq., GR $\to$ P present, *same* sign as ρ
adds to "active gravitational mass"
Q: why? Q: contrast with hydrostatic equilibrium?

Friedmann energy eq is "equation of motion" for scale factor
i.e., governs evolution of $a(t)$.

To solve, need to know how ρ depends on a

Density Evolution

1st Law of Thermodynamics (PS 2):

$$d(\varepsilon a^3) = -P d(a^3) \tag{3}$$

ε is energy density, and $\rho \equiv \varepsilon / c^2$ (mass-energy equivalence)

can solve for $\rho(a)$ given $P(\rho)$: equation of state
useful (though not most general) form:

$$P = w\varepsilon = w\rho c^2 \tag{4}$$

w is "equation of state parameter"
Q: w for matter? radiation?

for *constant* w:
$d(\rho a^3) = -w\rho d(a^3) \Rightarrow \boxed{\rho \propto a^{-3(1+w)}}$
Q: implications for matter? radiation?

Cosmic Constituents

In general:

$P = w\varepsilon = w\rho c^2 \Rightarrow$ $\boxed{\varepsilon = \rho c^2 \propto a^{-3(1+w)}}$

Matter (non-relativistic, a.k.a. "dust"):

$P_{\rm m} \ll \varepsilon_{\rm m} \approx \rho_{\rm m} c^2 \Rightarrow P_{\rm m} \simeq 0$ ($w_{\rm m} \simeq 0$)

$\Rightarrow \rho_{\rm m} \propto a^{-3}$

Radiation (relativistic species): $\gamma + 3\nu\bar{\nu}$ today

$P_{\rm rad} = \varepsilon_{\rm rad}/3 = 1/3\ \rho_{\rm rad} c^2 \Rightarrow w_{\rm rad} = 1/3$

$\rightarrow \rho_{\rm rad} \propto a^{-4}$

Cosmo constant (Λ, "vacuum energy"):

$P_\Lambda = -\varepsilon_\Lambda = -\rho_\Lambda c^2 \rightarrow w_\Lambda = -1$

$\rho_\Lambda = const$ (indep of a!) *Q: why is this bizarre?*

Q: if all these components exist, which dominates at late times? early times?

Cosmic Dynamic Epochs

Generally: $\rho_{\mathrm{tot}} = \rho_{\mathrm{m}} + \rho_{\mathrm{rad}} + \rho_{\Lambda}$
So: 3 (4?) major epochs
`www: Omega plot`

- Rad. dominated: $\rho_{\mathrm{tot}} \approx \rho_{\mathrm{rad}}$
  $\dot{a}/a \propto a^{-2} \Rightarrow a \propto t^{1/2}$

- Matter dominated: $\rho_{\mathrm{tot}} \approx \rho_{\mathrm{m}}$
  $\dot{a}/a \propto a^{-3/2} \Rightarrow a \propto t^{2/3}$

- Curvature dominated: ρ term $\ll$ curv. term ($\kappa = -1$)
  $\dot{a}/a \propto a^{-1} \Rightarrow a \propto t$

- Λ dominated:
  $\dot{a}/a = H = const \Rightarrow a \propto e^{Ht}$

Special Epochs: Changing Dominance

Matter, radiation, Λ, curvature
all appear in Friedmann with different a dependences
$\rightarrow$ dominant species trade off over time
$\rightarrow$ special epochs when one "overtakes" another

example: matter-radiation equality:
when $\rho_{\rm m} = \rho_{\rm rad}$
$\rho_{\rm rad} = \rho_{\rm rad,0} a^{-4}$
$\rho_{\rm m} = \rho_{\rm m,0} a^{-3}$
eq: $\rho_{\rm M} = \rho_{\rm rad}$
when $a_{\rm eq} = \rho_{\rm rad,0}/\rho_{\rm m,0} \sim 3 \times 10^{-4}$

Matter and Curvature

Consider a universe with matter and (maybe) curvature only

What if $\kappa = 1$?

$\Rightarrow$ positively curved $\rightarrow$ "spherelike" geometry

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-3} - \frac{c^2}{R^2}a^{-2}$$

a cannot grow without bound

Q: why? Q: what is $a_{\rm max}$?

Q: why are we sure that U recollapses after $t(a_{\rm max})$?

∞ fate: collapse continues back to $a = 0$: "big crunch!"

What if $\kappa = -1$?

$\Rightarrow$ "hyperbolic" geometry

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-3} + \frac{c^2}{R^2} a^{-2}$$

a grows without bound *Q: why?*

fate: expand forever–"big chill"

at large t, "curvature-dominated": $a(t) \to ct/R$ *Q: why?*

Q: how can we tell what our κ value is?

Geometry, Density, and Dynamics

rewrite Friedmann

$$1 = \frac{8\pi G\rho}{3H^2} - \frac{\kappa c^2}{R^2}(aH)^{-2} = \Omega - \frac{\kappa c^2}{R^2}(aH)^{-2} \tag{5}$$

where the **density parameter** is

$$\Omega = \frac{\rho}{\rho_{\rm crit}} \tag{6}$$

and the **critical density** is

$$\rho_{\rm crit} = \frac{3H^2}{8\pi G} \tag{7}$$

Note: for a particular density component ρ_i
corresponding density parameter is $\Omega_i = \rho_i/\rho_{\rm crit}$
and thus total sums all species: $\Omega \equiv \Omega_{\rm tot} = \sum_i \Omega_i$

Note that

$$\kappa = \left(\frac{aHR}{c}\right)^2 (\Omega - 1) = (\text{pos def}) \times (\Omega - 1)$$

fate* (and geometry) of Universe $\Leftrightarrow \kappa \Leftrightarrow \Omega - 1$

if $\Omega = 1$ ever:

- $\Omega = 1$ always; $\kappa = 0 \rightarrow$ expand forever

if $\Omega < 1$ ever:

- $\Omega < 1$ always; $\kappa = -1 \rightarrow$ expand forever

if $\Omega > 1$ ever:

- $\Omega > 1$ always; $\kappa = +1 \rightarrow$ recollapse

Q: but if Ω just a stand-in for κ, why useful?

*κ always gives geometry, but κ and fate decoupled if $\Lambda \neq 0$