Astro 596/496 NPA
Lecture 14
Sept. 25, 2009

Announcements:
- Problem Set 2 due
- Preflight 3 posted; due noon next Friday

Last time:
⋆ *most* cosmic mass-energy today is in dark energy!?!
⋆ *most* cosmic matter is in dark matter!?!

Surely this has implications for particle physics
Q: what properties must dark matter have?
what would this mean for particle dark matter?
Q: how about dark energy?

The Invisible Universe and Fundamental Physics

Dark Matter–what we know

- it exists
- is dark: can't have been detected yet
- is matter: $w_{\rm dm} \approx 0$

If DM is relic from early universe, DM particles must be

▷ stable (or long-lived)

▷ weakly interacting

▷ non-relativistic today

Good news:

particle theory offers many well-motivated DM candidates fitting this description

Dark Energy–what we know

- it exists
- is dark
- is energy, i.e.,$w < 0$

Major implications for fundamental physics:

need substance with $P \sim -\varepsilon$: pressure huge, negative!

but non-relativistic matter: $0 < w \ll 1$

relativistic matter: $w = 1/3$

$\rightarrow$ suggests any particle gas has $0 \leq w \leq 1/3$

Q: which means?

Bad news: particle theory taken by surprise!

no well-motivated dark energy candidates "off the shelf"

Good news: job security for cosmologists!

Cosmic Archaelogy: The Early Universe

is particle physics the key to the dark side?

When are high-energy processes/particles abundant?

- Universe has temperature now: CMB $T_0 = 2.725$ K
  $\Rightarrow$ cosmic matter was once in thermal equilibrium
- in thermal bath, typical particle energy is $E \sim kT$
- cosmic temperature $T \propto 1/a = 1 + z$

Therefore:

- when primordial soup at high-E $\longrightarrow$ high T $\longrightarrow$ early times

$\star$ the early universe is the realm of particle physics

$\star$ cosmic *particle* history $\Leftrightarrow$ cosmic *thermal* history

Cosmic Statistical Mechanics

Consider a “gas” of quantum particles (massive or massless) states “smeared out” around classical $\vec{x}$ and $\vec{p}$ values

define “occupation number” or “distribution function” $f(\vec{x},\vec{p})$: number of particles in each phase space “cell”:

$$dN = g f(\vec{x},\vec{p}) \, \frac{d^3\vec{x} \; d^3\vec{p}}{(2\pi\hbar)^3} = g f(\vec{x},\vec{p}) \, \frac{dV_{\rm space} \; dV_{\rm momentum}}{(2\pi\hbar)^3} \qquad (1)$$

where g counts internal (spin/helicity) degrees of freedom and $dx\,dp/2\pi\hbar$ counts # of quantum states per cell

particle phase space occupation f determines bulk properties

Q: how? Hint–what’s # particles per unit spatial volume?

for a given spatial volume element $dV_{\text{space}} = d^3\vec{x} = dx\,dy\,dz$
number per unit (spatial) volume–i.e., **number density**–is

$$dn = \frac{dN}{d^3\vec{x}} g\, f(\vec{x},\vec{p})\ \frac{d^3\vec{p}}{(2\pi\hbar)^3} \tag{2}$$

$\longrightarrow$ f gives *distribution* of momenta at each spatial point

Q: what's f for gas of (classical) particles all at rest?
Q: f for a (classical) particle beam–directed, monoenergetic?
Q: what's f for (classical) harmonic oscillator ensemble?

Q: given f, how to formally compute bulk properties n, ε, P?

Number density

$$n(\vec{x}) = \frac{d^3N}{d^3x} = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ f(\vec{p},\vec{x}) \tag{3}$$

Mass-energy density

$$\varepsilon(\vec{x}) = \rho(\vec{x})c^2 = \langle E \rangle\ n = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ E(p)\ f(\vec{p},\vec{x}) \tag{4}$$

Pressure

$$P(\vec{x}) = \langle p_i v_i \rangle_{\text{direction}\,i}\ n = \frac{\langle pv \rangle}{3} n = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ \frac{p\,v(p)}{3}\ f(\vec{p},\vec{x}) \tag{5}$$

Q: these expressions are general–simplifications in FLRW?

FRLW universe:

- homogeneous $\rightarrow$ no $\vec{x}$ dep
- isotropic $\rightarrow$ only $\vec{p}$ magnitude important $\rightarrow$ $f(\vec{p}) = f(p)$

in **thermal equilibrium**:

▷ Boson occupation number is **Bose-Einstein** dist'n

$$f_{\rm b}(p) = \frac{1}{e^{(E-\mu)/kT} - 1} \tag{6}$$

▷ Fermion occupation number is **Fermi-Dirac** dist'n

$$f_{\rm f}(p) = \frac{1}{e^{(E-\mu)/kT} + 1} \tag{7}$$

Note: μ is "chemical potential" or "Fermi energy"
$\mu = \mu(T)$ but is *independent* of E

∞

If $E = E_{\rm tot}, \mu \gg T$: both $\rightarrow$ $f = e^{-(E-\mu)/kT} \ll 1$
$\rightarrow$ **Boltzmann distribution**

Chemical Potential & Number Conservation

For a particle species in thermal equilibrium

$$f(p;T,\mu) = \frac{1}{e^{(E-\mu)/kT} \pm 1} \tag{8}$$

What is μ, and what does it mean physically?

First, what if $\mu = 0$
then f, n, P depend only on T
$\longrightarrow$ everything at same T has same ρ, P!
sometimes true! *Q: examples?* but not always!

but n often conserved
$\longrightarrow$ fixed by initial conditions, not T
$\longrightarrow$ if particle number conserved, μ determined
by solving $n_{\text{cons}} = n(\mu, T) \rightarrow \mu(n_{\text{cons}}, T)$
so: $\boxed{\mu \neq 0 \Leftrightarrow \text{particle number conservation}}$

if "chemical" equilibrium:

- rxns change particle numbers among species
- equilibrium: forward rate = reverse rate

$a + b + \cdots \leftrightarrow A + B + \cdots$

then

$$\sum_{\text{initial particles} i} \mu_i = \sum_{\text{final particles} f} \mu_f \tag{9}$$

sum of chemical potentials "conserved"

Radiation and Matter: The Life of a Particle Species

A given type of particle can act
sometimes as cosmic *matter*
other times as cosmic *radiation*

Q: criteria?

Q: radiation species today?
Q: when did this list last change? when before that?

Equilibrium Thermodynamics

Gas of mass m particles at temp T:
n, ρ, and P in general complicated
because of $E(p) = \sqrt{p^2 + m^2}$
but simplify in ultra-rel and non-rel limits
$\longrightarrow$ controlled by m vs T comparison

Non-Relativistic Species

$E(p) \simeq mc^2 + p^2/2m$, $T \ll m$
for $\mu \ll T$: Maxwell-Boltzmann, same for Boson, Fermions

for non-relativistic particles = *matter*
energy density, number density vs T?

Non-Relativistic Species: Cosmic Matter

In the limit $E(p) \simeq mc^2 + p^2/2m$, $T \ll m$

$$n = g \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} e^{-(mc^2-\mu)/kT} \tag{10}$$

$$\rho c^2 = mc^2\, n + \frac{3}{2}\, kT\, n \simeq \varepsilon_{\text{rest mass}} = mc^2\, n \tag{11}$$

$$P = \frac{2}{3}\varepsilon_{\text{kinetic}} = nkT \ll \rho c^2 \tag{12}$$

Note:

- recover ideal gas law!
- $P \ll \rho c^2 \rightarrow w_{\text{non-rel}} \ll 1 \approx 0$
- if particles *not* conserved: $\mu = 0$
  Q: behavior of $n(T)$? why isn't this crazy?
- if particles *are conserved*: $\mu(T) \neq 0$

$\longrightarrow$ this sets number density implicitly
  i.e., $n(T, \mu) = n_{\text{cons}}$ sets values of μ
  in cosmo setting: $n_{\text{non-rel,cons}} \propto a^{-3}$, $\rho_{\text{non-rel,cons}} \simeq mn \propto a^{-3}$

Ultra-Relativistic Species: Cosmic Radiation

take limit $E(p) \simeq cp \gg mc^2$ (i.e., $kT \gg mc^2$)

Also take $\mu = 0$ ($\mu \ll kT$)

note: now contributions from states with $E, \mu \ll T$
expect bosons, fermions $\rightarrow$ different n, ρ, P for same T
Q: why? Hint–think about form of $f_{\rm b}$ *and* $f_{\rm f}$
Q: which particle type should have larger n, ρ, P *at fixed* T*?*

energy density, number density?
Q: you know this already for bosons!

for relativistic bosons

$$n_{\rm rel,b} = g\frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \propto T^3$$
$$\rho_{\rm rel,b}c^2 = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \propto T^4$$

where

$$\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.20206\ldots \quad (13)$$

relativistic fermions:

$$n_{\rm rel,f} = \frac{3}{4}n_{\rm rel,b} \quad (14)$$
$$\rho_{\rm rel,f} = \frac{7}{8}\rho_{\rm rel,b} \quad (15)$$

so $n \propto T^3$ and $\rho \propto T^4$ for both

e.g., CMB today: $n_{\gamma,0} = 411\ \mathrm{cm}^{-3}$

also $n_{\rm rel,f} < n_{\rm rel,b}$ and $\rho_{\rm rel,f} < \rho_{\rm rel,b}$ (Pauli)

For both relativistic bosons and fermions (with $\mu \ll T$):

$$P_{\rm rel} = \frac{1}{3}\rho_{\rm rel} c^2 \tag{16}$$

★ holds for *both* fermions and bosons!

e.g., $P_{\rm rel,f} = \rho_{\rm rel,f}/3 < P_{\rm rel,b}$

★ shows that relativistic particles have $w_{\rm rel} = +1/3$

★ $P \propto T^4$

Radiation Evolution

Cosmic radiation density sums over relativistic species:

$$\rho_{\rm rel} = \sum_i \rho_{{\rm rel},i} \tag{17}$$

$$= \frac{\pi^2}{30} T^4 \left[\sum_{\rm bosons} g_{\rm b} \left(\frac{T_{\rm b}}{T} \right)^4 + \sum_{\rm fermions} \frac{7}{8} g_{\rm f} \left(\frac{T_{\rm f}}{T} \right)^4 \right] \tag{18}$$

$$= g_*(T) \frac{\pi^2}{30} T^4 \tag{19}$$

where

- T is for some reference species, usually photons
- g_* counts "relativistic degrees of freedom"
  e.g., photons contribute $g_{*,\gamma} = 2$
  left-handed $\nu\bar{\nu}$ contributes $g_{*,\nu} = 2 \cdot 7/8 = 7/4$

Particle Census and the Radiation Era

In radiation-dominated early universe:

$$\left(\frac{\dot{a}}{a}\right)^2 \approx 8\pi G \rho_{\text{rel}}/3 \propto g_*(T)\ T^4 \tag{20}$$

- ⋆ early expansion history depends on number, types of relativistic particles
- ⋆ microphysics (particle content) of the Universe controls macroscopic cosmic dynamics
- ⋆ ...so any *measure* of early expansion rate is a probe of particle physics!
  ... as we will soon see

Director's Cut Extras

Kinetic Theory of Pressure due to Particle Motions

consider cubic box, sidelength L (doesn't really need to be cubic)
contain "gas" of N particles: can be massive or massless
particles collide with walls, bounce back elastically
particles exert force on wall $\leftrightarrow$ wall on particles
this lead to bulk *pressure*

focus on one particle, and its component of motion
in one (arbitrary) axis x: speed v_x, momentum p_x

- *elastic* collision: $p_{x,init} = -p_{x,fin} \rightarrow \delta p_x = 2p_x$
- collision time interval for same wall: $\delta t_x = v_x/2L$
- single-particle *momentum transfer* (force) per wall:
  $F_x = \delta p_x/\delta t_x = p_x v_x/L$
- *single-particle force* per wall area:
  $P = F_x/L^2 = p_x v_x/L^3 = p_x v_x/V$

Q: total pressure?

total pressure is sum over all particles:

$$P = \sum_{\text{particles } \ell=1}^{N} \frac{p_x^{(\ell)} v_x^{(\ell)}}{V} \tag{21}$$

can rewrite in terms of an average momentum flux

$$P = \frac{N}{V} \frac{\sum_{\ell=1}^{N} p_x^{(\ell)} v_x^{(\ell)}}{N} = \langle p_x v_x \rangle n \tag{22}$$

where $n = N/V$ is *number* density
$\langle p_x \rangle n$ would be average *momentum density* along x
and $\langle p_x v_x \rangle n$ is average *momentum flux* along x

if particle gas has isotropic momenta, then

$$\langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_x \rangle = \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle = \frac{1}{3} \langle p v \rangle \tag{23}$$

so $\boxed{P = \frac{1}{3} \langle p v \rangle n}$

Temperature Evolution Revisited

If in therm eq, maintain photon occ. #

$$f(p) = \frac{1}{e^{p/T} - 1} \tag{24}$$

but $cp = h\nu = hc/\lambda \propto 1/a(t)$:
$\Rightarrow p = p_0/a$

w/o interactions, const # γ per mode p
$\Rightarrow f(p) = const$
$\Rightarrow p(t)/T(t) = p_0/T_0$
$\Rightarrow \boxed{T/T_0 = p/p_0 = 1/a = 1 + z}$
e.g., at $z = 3$, CMB $T = 4T_0 \simeq 11$ K
(measured in QSO absorption line system!)

recall: used $w = 1/3$ to show $\rho_\gamma \propto a^{-4}$
but blackbody $\rho_\gamma \propto T^4$
together $T \propto 1/a$ (OK!)