Announcements:
- Problem Set 2 due
- Preflight 3 posted; due noon next Friday

Last time:
★ *most* cosmic mass-energy today is in dark energy!?!
★ *most* cosmic matter is in dark matter!?!

Surely this has implications for particle physics

Q: *what properties must dark matter have?*

what would this mean for particle dark matter?

Q: *how about dark energy?*
The Invisible Universe and Fundamental Physics

**Dark Matter**—what we know

- it exists
- is dark: can’t have been detected yet
- is matter: $w_{\text{dm}} \approx 0$

*If* DM is relic from early universe, DM particles must be

- stable (or long-lived)
- weakly interacting
- non-relativistic today

*Good news:*

particle theory offers many well-motivated DM candidates fitting this description
Dark Energy—what we know

• it exists
• is dark
• is energy, i.e., $w < 0$

Major implications for fundamental physics:
need substance with $P \sim -\varepsilon$: pressure huge, negative!
  but non-relativistic matter: $0 < w \ll 1$
  relativistic matter: $w = 1/3$
→ suggests any particle gas has $0 \leq w \leq 1/3$
  $Q$: which means?

Bad news: particle theory taken by surprise!
  no well-motivated dark energy candidates “off the shelf”

Good news: job security for cosmologists!
Cosmic Archaeology: The Early Universe

is particle physics the key to the dark side?

When are high-energy processes/particles abundant?
• Universe has temperature now: CMB $T_0 = 2.725$ K
  ⇒ cosmic matter was once in thermal equilibrium
• in thermal bath, typical particle energy is $E \sim kT$
• cosmic temperature $T \propto 1/a = 1 + z$

Therefore:
• when primordial soup at high-$E$ → high $T$ → early times

★ the early universe is the realm of particle physics
★ cosmic particle history ⇔ cosmic thermal history
Cosmic Statistical Mechanics

Consider a “gas” of quantum particles (massive or massless) states “smeared out” around classical \( \vec{x} \) and \( \vec{p} \) values
define “occupation number” or “distribution function” \( f(\vec{x}, \vec{p}) \):
number of particles in each phase space “cell”:

\[
dN = g f(\vec{x}, \vec{p}) \frac{d^3\vec{x} \, d^3\vec{p}}{(2\pi\hbar)^3} = g f(\vec{x}, \vec{p}) \frac{dV_{\text{space}} \, dV_{\text{momentum}}}{(2\pi\hbar)^3} \tag{1}
\]

where \( g \) counts internal (spin/helicity) degrees of freedom and \( dx \, dp/2\pi\hbar \) counts \# of quantum states per cell

particle phase space occupation \( f \) determines bulk properties

Q: how? Hint—what’s \# particles per unit spatial volume?
for a given spatial volume element $dV_{\text{space}} = d^3\vec{x} = dx
dy\,dz$
number per unit (spatial) volume—i.e., **number density**—is

$$dn = \frac{dN}{d^3\vec{x}} g f(\vec{x}, \vec{p}) \frac{d^3\vec{p}}{(2\pi\hbar)^3}$$  \hspace{1cm} (2)

→ $f$ gives *distribution* of momenta at each spatial point

**Q:** what’s $f$ for gas of (classical) particles all at rest?
**Q:** $f$ for a (classical) particle beam–directed, monoenergetic?
**Q:** what’s $f$ for (classical) harmonic oscillator ensemble?

**Q:** given $f$, how to formally compute
  *bulk properties* $n$, $\varepsilon$, $P$?
Number density

\[ n(\vec{x}) = \frac{d^3 N}{d^3 x} = \frac{g}{(2\pi \hbar)^3} \int d^3 \vec{p} \ f(\vec{p}, \vec{x}) \] (3)

Mass-energy density

\[ \varepsilon(\vec{x}) = \rho(\vec{x}) c^2 = \langle E \rangle \ n = \frac{g}{(2\pi \hbar)^3} \int d^3 \vec{p} \ E(p) \ f(\vec{p}, \vec{x}) \] (4)

Pressure

\[ P(\vec{x}) = \langle p_i v_i \rangle_{\text{direction}} n = \frac{\langle p v \rangle}{3} \ n = \frac{g}{(2\pi \hbar)^3} \int d^3 \vec{p} \ \frac{p v(p)}{3} \ f(\vec{p}, \vec{x}) \] (5)

Q: these expressions are general–simplifications in FLRW?
FRLW universe:
• homogeneous $\rightarrow$ no $\vec{x}$ dep
• isotropic $\rightarrow$ only $\vec{p}$ magnitude important $\rightarrow f(\vec{p}) = f(p)$

in thermal equilibrium:
▷ Boson occupation number is Bose-Einstein dist'n
\[
f_b(p) = \frac{1}{e^{(E-\mu)/kT} - 1} \tag{6}
\]
▷ Fermion occupation number is Fermi-Dirac dist'n
\[
f_f(p) = \frac{1}{e^{(E-\mu)/kT} + 1} \tag{7}
\]

Note: $\mu$ is “chemical potential” or “Fermi energy” $\mu = \mu(T)$ but is independent of $E$

If $E = E_{\text{tot}}, \mu \gg T$: both $\rightarrow f = e^{-(E-\mu)/kT} \ll 1$
$\rightarrow$ Boltzmann distribution
Chemical Potential & Number Conservation

For a particle species in thermal equilibrium

\[ f(p; T, \mu) = \frac{1}{e^{(E-\mu)/kT} \pm 1} \]  

What is \( \mu \), and what does it mean physically?

First, what if \( \mu = 0 \)
then \( f, n, P \) depend only on \( T \)
\( \rightarrow \) everything at same \( T \) has same \( \rho, P! \)
sometimes true! Q: examples? but not always!

but \( n \) often conserved
\( \rightarrow \) fixed by initial conditions, not \( T \)
\( \rightarrow \) if particle number conserved, \( \mu \) determined
by solving \( \n_{\text{cons}} = n(\mu, T) \rightarrow \mu(n_{\text{cons}}, T) \)
so: \( \mu \neq 0 \iff \text{particle number conservation} \)
if “chemical” equilibrium:
• rxns change particle numbers among species
• equilibrium: forward rate = reverse rate

\[ a + b + \cdots \leftrightarrow A + B + \cdots \]

then

\[ \sum_{\text{initial particles}} \mu_i = \sum_{\text{final particles}} \mu_f \]  

(9)

sum of chemical potentials “conserved”
Radiation and Matter: The Life of a Particle Species

A given type of particle can act sometimes as cosmic matter other times as cosmic radiation

Q: criteria?

Q: radiation species today?
Q: when did this list last change? when before that?
Equilibrium Thermodynamics

Gas of mass $m$ particles at temp $T$:
n, $\rho$, and $P$ in general complicated
because of $E(p) = \sqrt{p^2 + m^2}$
but simplify in ultra-rel and non-rel limits
→ controlled by $m$ vs $T$ comparison

Non-Relativistic Species

$E(p) \simeq mc^2 + p^2/2m, \ T \ll m$
for $\mu \ll T$: Maxwell-Boltzmann, same for Boson, Fermions

for non-relativistic particles = matter
energy density, number density vs $T$?
Non-Relativistic Species: Cosmic Matter

In the limit $E(p) \simeq mc^2 + p^2/2m$, $T \ll m$

\[
n = g \left( \frac{m k T}{2 \pi \hbar^2} \right)^{3/2} e^{-(mc^2-\mu)/kT} \tag{10}
\]

\[
\rho c^2 = mc^2 n + \frac{3}{2} kT n \simeq \varepsilon_{\text{rest mass}} = mc^2 n \tag{11}
\]

\[
P = \frac{2}{3} \varepsilon_{\text{kinetic}} = nkT \ll \rho c^2 \tag{12}
\]

Note:
- recover ideal gas law!
- $P \ll \rho c^2 \rightarrow w_{\text{non-rel}} \ll 1 \approx 0$
- if particles not conserved: $\mu = 0$
  - Q: behavior of $n(T)$? why isn't this crazy?
- if particles are conserved: $\mu(T) \neq 0$
  - this sets number density implicitly
  - i.e., $n(T, \mu) = n_{\text{cons}}$ sets values of $\mu$
  - in cosmo setting: $n_{\text{non-rel,cons}} \propto a^{-3}$, $\rho_{\text{non-rel,cons}} \simeq mn \propto a^{-3}$
Ultra-Relativistic Species: Cosmic Radiation

Take limit $E(p) \simeq cp \gg mc^2$ (i.e., $kT \gg mc^2$)

Also take $\mu = 0$ ($\mu \ll kT$)

Note: now contributions from states with $E, \mu \ll T$

Expect bosons, fermions $\rightarrow$ different $n, \rho, P$ for same $T$

Q: why? Hint–think about form of $f_b$ and $f_f$

Q: which particle type should have larger $n, \rho, P$ at fixed $T$?

Energy density, number density?

Q: you know this already for bosons!
for relativistic bosons

\[ n_{\text{rel,b}} = g \frac{\zeta(3)}{\pi^2} \left( \frac{kT}{\hbar c} \right)^3 \propto T^3 \]

\[ \rho_{\text{rel,b}} c^2 = g \frac{\pi^2}{30} \left( \frac{kT}{\hbar c} \right)^4 \propto T^4 \]

where

\[ \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.20206 \ldots \] (13)

relativistic fermions:

\[ n_{\text{rel,f}} = \frac{3}{4} n_{\text{rel,b}} \] (14)

\[ \rho_{\text{rel,f}} = \frac{7}{8} \rho_{\text{rel,b}} \] (15)

so \( n \propto T^3 \) and \( \rho \propto T^4 \) for both

e.g., CMB today: \( n_{\gamma,0} = 411 \text{ cm}^{-3} \)

also \( n_{\text{rel,f}} < n_{\text{rel,b}} \) and \( \rho_{\text{rel,f}} < \rho_{\text{rel,b}} \) (Pauli)
For both relativistic bosons and fermions (with $\mu \ll T$): 

$$P_{\text{rel}} = \frac{1}{3} \rho_{\text{rel}} c^2$$ 

(16)

★ holds for both fermions and bosons!

e.g., $P_{\text{rel},f} = \rho_{\text{rel},f}/3 < P_{\text{rel},b}$

★ shows that relativistic particles have $w_{\text{rel}} = +1/3$

★ $P \propto T^4$
Radiation Evolution

Cosmic radiation density sums over relativistic species:

\[
\rho_{\text{rel}} = \sum_i \rho_{\text{rel},i} \tag{17}
\]

\[
= \frac{\pi^2}{30} T^4 \left[ \sum_{\text{bosons}} g_b \left( \frac{T_b}{T} \right)^4 + \sum_{\text{fermions}} \frac{7}{8} g_f \left( \frac{T_f}{T} \right)^4 \right] \tag{18}
\]

\[
= g^* \left( T \right) \frac{\pi^2}{30} T^4 \tag{19}
\]

where

- \( T \) is for some reference species, usually photons
- \( g^* \) counts “relativistic degrees of freedom”
  - e.g., photons contribute \( g^*, \gamma = 2 \)
  - left-handed \( \nu \bar{\nu} \) contributes \( g^*, \nu = 2 \cdot \frac{7}{8} = \frac{7}{4} \)
Particle Census and the Radiation Era

In radiation-dominated early universe:

\[
\left( \frac{\dot{a}}{a} \right)^2 \approx \frac{8\pi G \rho_{\text{rel}}}{3} \propto g^* (T') T^4
\]  (20)

- early expansion history depends on number, types of relativistic particles
- microphysics (particle content) of the Universe controls macroscopic cosmic dynamics
- ...so any measure of early expansion rate is a probe of particle physics!
  ... as we will soon see
Kinetic Theory of Pressure due to Particle Motions

consider cubic box, sidelength $L$ (doesn’t really need to be cubic)
contain “gas” of $N$ particles: can be massive or massless
particles collide with walls, bounce back elastically
particles exert force on wall $\leftrightarrow$ wall on particles
this lead to bulk pressure

focus on one particle, and its component of motion
in one (arbitrary) axis $x$: speed $v_x$, momentum $p_x$

- elastic collision: $p_{x,\text{init}} = -p_{x,\text{fin}} \rightarrow \delta p_x = 2p_x$
- collision time interval for same wall: $\delta t_x = v_x/2L$
- single-particle momentum transfer (force) per wall:
  $$F_x = \frac{\delta p_x}{\delta t_x} = \frac{p_x v_x}{L}$$
- single-particle force per wall area:
  $$P = \frac{F_x}{L^2} = \frac{p_x v_x}{L^3} = \frac{p_x v_x}{V}$$

Q: total pressure?
total pressure is sum over all particles:

\[ P = \sum_{\text{particles } \ell=1}^{N} \frac{p_x^{(\ell)} v_x^{(\ell)}}{V} \]  \hspace{1cm} (21)

can rewrite in terms of an average momentum flux

\[ P = \frac{N}{V} \sum_{\ell=1}^{N} \frac{p_x^{(\ell)} v_x^{(\ell)}}{N} = \langle p_x v_x \rangle n \]  \hspace{1cm} (22)

where \( n = N/V \) is number density
\( \langle p_x \rangle n \) would be average momentum density along \( x \)
and \( \langle p_x v_x \rangle n \) is average momentum flux along \( x \)

if particle gas has isotropic momenta, then

\[ \langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_x \rangle = \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle = \frac{1}{3} \langle pv \rangle \]  \hspace{1cm} (23)

so

\[ P = \frac{1}{3} \langle pv \rangle n \]
Temperature Evolution Revisited

If in therm eq, maintain photon occ. #

\[ f(p) = \frac{1}{e^{p/T} - 1} \]  

(24)

but \( cp = h\nu = \frac{hc}{\lambda} \propto 1/a(t) \):

\[ \Rightarrow p = \frac{p_0}{a} \]

w/o interactions, const # \( \gamma \) per mode \( p \)

\[ \Rightarrow f(p) = const \]

\[ \Rightarrow p(t)/T(t) = p_0/T_0 \]

\[ \Rightarrow \frac{T}{T_0} = \frac{p}{p_0} = \frac{1}{a} = 1 + z \]

e.g., at \( z = 3 \), CMB \( T = 4T_0 \approx 11 \) K

(measured in QSO absorption line system!)

recall: used \( w = 1/3 \) to show \( \rho_\gamma \propto a^{-4} \)

but blackbody \( \rho_\gamma \propto T^4 \)
together \( T \propto 1/a \) (OK!)

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