Astronomy 350 Fall 2011
Homework #6

Due in class: Friday, Oct. 14

1. *Length Contraction.* In class we considered a light clock on its side. This a traincar whose length is $L_{\rm rest}$ when measured by an observer at rest in the traincar. A light pulse that travels from back to front and back again. And observer in the car times the roundtrip time of the light pulse, which we will call $t_{\rm rest}$. All the while, the train is moving at speed v relative to a bystander. The bystander measures the roundtrip time for the light pulse to be $t_{\rm moving}$.

    (a) [**5 points**]. Show that $t_{\rm rest} = 2L_{\rm rest}/c$.

    For the case of the bystander who sees the train as moving, notice that the light pulse appears to go faster than the train by speed $c - v$ when it goes back-to-front, and after reflection appears to move at speeed $c - v$ towards the back. Using this information, show that

    $$t_{\rm moving} = \frac{L_{\rm moving}}{c - v} + \frac{L_{\rm moving}}{c + v} \tag{1}$$

    where $L_{\rm moving}$ is the the apparent distance the light travels.

    (b) [**5 points**]. Finally, use the results from part (a), as well as the time dilation formula $t_{\rm moving} = t_{\rm rest}/\sqrt{1 - v^2/c^2}$, to verify that $L_{\rm moving} = \sqrt{1 - v^2/c^2}\ L_{\rm rest}$.

2. *Special Relativity: Length Contraction and Time Dilation in the Real World.* Special relativity predicts strange effects when one observer measures the behavior of clocks and yardsticks of another observer in relative motion. These results are not effects we are aware of in everyday life; here we will see why.

    In all parts of this problem, the situation is that there are two observers, A and B, in relative motion with constant velocity. Their relative speed is v and along the x-axis. You should ignore effects of gravity, i.e., this is a special relativity problem only. Each observer has a yardstick of length $L_{\rm rest} = 1$ yard (as measured by the observer her/himself) which lies along the x-axis, and a clock that ticks once every $(\Delta t)_{\rm rest} = 1$ second (as measured by the observer her/himself).

    (a) [**4 points**]. Assign names to the observers A and B. Use these throughout the rest of the problem.

    First, consider a relative speed v which is familiar from everyday experience, say $v = 65$ mph $= 30$ meters/sec. A observes B moving with this speed. Calculate the length $L_{\rm obs}$ and tick rate $(\Delta t)_{\rm obs}$ that A observes B's instruments to have. Also calculate the same results that B observes A's instruments to have.

    Note that your calculator may have trouble here, giving a perhaps surprising answer. If so, explain where the problem crops up.

    (b) [**4 points**]. Now consider a faster speed: $v = 200$ km/sec (about the Sun's orbit speed around the center of the Galaxy). As in part (a), find A's impressions of B's instruments and vice versa.

    At this much higher speed, how noticeable are the relativistic effects?

(c) [**4 points**]. Finally, consider a speed $v = 99\%c$. Again, find A's impressions of B's instruments and vice versa. Comment on your result.

(d) [**4 points**]. On the basis of your answers from parts (a)-(c), come up with a "rule of thumb" for when (i.e., at what speeds v) weird relativity effects are noticeable, and when they are not. Go on to explain why in everyday life we don't notice length contraction and time dilation.

3. *Special Relativity: Energy.*

   (a) [**4 points**]. Compute the rest energy of a donut with mass $m = 25$ grams $= 0.025$ kg. Your result should be in units of Joules, where 1 Joule $= 1$ kg m^2/s^2; don't be surprised if the answer is a large number!

   To get a feel for this result, let's compare it to other more familiar energies.

      i. Compare the donut rest energy to the gravitational potential energy of the same donut located on the Earth's surface (i.e., a distance $R = 6400$ km from the Earth's center).

      ii. Compare the donut rest energy to the chemical energy in the donut, i.e., the energy available in food calories. Note that a typical donut has, say, 200 Calories, where 1 Calorie $= 4000$ Joules.

      iii. If a donut's rest energy were to power a lightbulb, how long would the light say lit? Take the light bulb's power requirement (energy loss per second) to be $L = 100$ Watts $= 100$ Joules/sec. Find the "lit time" in seconds and then in years.

   Comment on these results.

   (b) [**4 points**]. The donut rest energy is clearly enormous. Why don't we notice this in everyday life? What would it take to liberate this energy?

   (c) [**4 points**]. Consider a particle of mass m moving (relative to you) at speeds very close to c. Find the particle energy E for $v = 0.99c$, then for $v = 0.999c$, then for $v = 0.9999c$. You may express your answer in terms of mc^2, i.e., you may report your answers as $E = X\ mc^2$, where you give the value for X. Comment on the implications of your results for acceleration of particles (or spacecraft!) to high speeds.

   (d) [**4 points**]. At Fermilab, protons are accelerated until each has a total energy $E = 1000 m_{\text{proton}} c^2$. Find the speed of a proton with this energy. You may express your result as a fraction of the speed of light, i.e., you may solve for v/c. Comment on your result.

4. *Special Relativity: the Non-Relativistic Limit.* In class, it was alleged that the relativistic energy expression for a particle of mass m and speed v is well-approximated, for slow speeds $v \ll c$ but $v \neq 0$, by the expression

$$E_{\text{approx}} \approx mc^2 + \frac{1}{2}mv^2 \tag{2}$$

   Here we will verify that this is true.

(a) [**4 points**]. First, show that the above approximation is exactly true for the case $v = 0$. Use this result to interpret the first term in the approximation.

Then go on to interpret the second term in the approximation.

(b) [**4 points**]. One way of showing that this approximation is a good one is to show that the "generalized kinetic energy" $KE = E_{\text{total}} - mc^2$ becomes $\frac{1}{2}mv^2$ for slow speeds (here E_{total} is Einstein's full, exact Special Relativity expression for total energy). So the problem amounts to finding $KE = E_{\text{total}} - mc^2$ for $v \ll c$ but $v \neq 0$. To do this, you can use the mathematical result that

$$(1 - x)^{-1/2} \approx 1 + \frac{x}{2} \qquad (3)$$

for $x \ll 1$. Using this, you should be able to show that $KE \approx \frac{1}{2}mv^2$ for small speeds.

Briefly explain why it is essential for Relativity that your result from (b) holds.

(c) [**4 bonus points**]. Show that equation (3) holds. To do this, either (i) take the technical route, and use a Taylor/Maclaurin expansion, or (ii) try different values of x in your calculator and show that for small x the result works well.