**Astronomy 406, Fall 2013**
**Problem Set #1**

Due in class: Friday, Sept. 6   Total Points: 60 + 5 bonus

*Note:* Your homework solutions should be legible and include all calculations, diagrams, and explanations. The TA is not responsible for deciphering unreadable or illegible problem sets! Also, homework is graded on the method of solution, not just the final answer; you may not get any credit if you just state the final answer!

You may discuss with other students, but you are responsible for your own answers: you must understand your solutions, and you must write them yourself *in your own words.*

1. *Keplerian Motion Post-Summer Warmup.* Consider a particle $m$ moving in the gravity field set up by much larger spherical mass $M$. The Newtonian gravitational force on $m$ has magnitude $F = GMm/r^2$. The particle moves in a circular orbit at constant distance $r = a$.

   (a) [4 points] Find the circular speed $v_c$. Recall that the centripetal acceleration $a_c = v_c^2/r$ and is provided by the gravitational acceleration.

   (b) [4 points] Let $P$ be the orbital period. Show that the orbit obeys Kepler's third law, $a^3 \propto P^2$, and find an expression for the constant of proportionality. Go on to solve for $P$.

   (c) [4 points] Show that the orbit always obeys the relationship $2T = -U$, where $T = 1/2\ mv^2$ is particle the kinetic energy, and $U = -GMm/r$ is the potential energy. Then show how the total energy $E_{\rm tot}$ is related to $U$. (We will soon see that these are special cases of a general result known as the Virial theorem.)

2. *Gravitational Timescales from Dimensional Analysis.* Throughout this course we will often consider systems in which gravity is the only or the dominant driver of the system's motion. In such cases, it is useful to calculate or at least estimate the characteristic timescale for the system to evolve given its present state (mass distribution). It turns out that just from dimensional analysis one can arrive at a very useful "back-of-the-envelope" estimate for the gravitational timescale. This problem is wordy but not very difficult, and will be useful throughout the course.

   (a) [4 points] Consider a system with a mass $m$ and a characteristic lengthscale $\ell$. Find the physical dimensions (i.e., factors of length, time, and mass) in Newton's gravitational constant $G$. Using these, show that you can construct an algebraic combination of $G$, $m$, and $\ell$ which has the dimensions of time. Call this expression $\tau_{\rm grav}$, the gravitational timescale. Go on to show that this combination can be very compactly expressed in terms of the system's characteristic mass density $\rho \sim m/\ell^3$. Note that for a dimensional analysis like this we include only terms with physical dimensions and not dimensionless numerical prefactors (like 2, or $\pi$, etc).

   Briefly comment on the uniqueness of your expression–what are other combination (if any) of the factors of $G$, $m$, and $\ell$ that also give a timescale?

   (b) [4 points] For the circular-motion Kepler problem you solved in Question 1 above, identify the appropriate mass and length scales $m$ and $\ell$. Using these, use your expression from part (a) to compute $\tau_{\rm grav}$. What timescale does this physically correspond to? How accurate is your estimate compared to the full result from Problem 1? For the Kepler problem, what is the physical interpretation of the characteristic density scale $\rho$?

(c) [5 points] Now consider a spherical matter distribution of radius $R$ and (constant) density $\rho$. Assume this object had no other forces, i.e., suppose gas pressure and/or solid-state interactions were insignificant or "turned off." Then in the absence of opposing forces, the object will collapse under its own gravity. Find $\tau_{\rm grav}$ for this system. To see that this is a reasonable result, make a different estimate for the collapse timescale by asking how long it would take a particle at the surface of the sphere to fall to the center, assuming it always is accelerated by a constant gravitational acceleration $g$, which you may take to be the pre-collapse acceleration at $R$. How does your result compare with $\tau_{\rm grav}$?

Finally, evaluate the gravitational timescale for the Earth (mean density $\rho_{\rm Earth} \approx 5.5\ {\rm g\,cm^{-3}}$), the Sun (mean density $\rho_\odot \approx 1.4\ {\rm g\,cm^{-3}}$), and a neutron star (mean density $\rho_\odot \approx 10^{15}\ {\rm g\,cm^{-3}}$). Comment on your results. If you find that the timescale is shorter than the known ages of these objects (billions of years), explain the discrepancy.

(d) [5 **bonus** points] Now consider a sphere of *non*-uniform density, also undergoing gravitational collapse. If, as in most stars, the density *decreases* with radius, from a maximum at the center to a minimum at the surface, how would you expect the collapse to proceed? Don't do any calculation here, but use the form of $\tau_{\rm grav}$ to guide your reasoning. Also, what if the density were to *increase* with radius from a minimum at the center to a maximum at the surface?

(e) [4 points] Now consider the Universe itself. We will see that the cosmic mass density today is about $\rho_0 = 3 \times 10^{-30}\ {\rm g\,cm^{-3}}$. Evaluate the associated gravitational timescale. Express your answer in seconds and in billions of years ($10^9$ yr $= 1$ Gyr). What might be the physical significance of this timescale?

3. *Flux and the inverse square law* [5 points] The Sun's flux as seen at Earth is $F_\odot = 1360\ {\rm Watt/m^2}$; this is sometimes called the "solar constant." Calculate how far away you need to place a 100 Watt lightbub for it to have the same flux as the Sun's. Does this result seem reasonable?

   The star with the next largest flux after the Sun is Sirius (the "dog star"), which has, at visible wavelengths, about $F_{\rm Sirius} = 7.7 \times 10^{-11}\ F_\odot$. Calculate how far away you need to place a 100 Watt lightbub for it to have the same flux as Sirius. Does this result seem reasonable?

4. *How far can the eye see?*

(a) [4 points] The magnitude scale is the traditional and somewhat confusing way astronomers measure brightness or flux. The rule is that the *apparent* magnitude $m$ of a star is related to its flux $F$ by

$$m = -2.5 \log \left( \frac{F}{F_{\rm zp}} \right) \tag{1}$$

where the factor of 2.5 is arbitrary, and the sign is a huge source of confusion, as it means that brighter stars have smaller magnitudes. Sorry. $F_{\rm zp}$ is a fixed (arbitrary) reference flux.

Show that when $F = F_{\rm zp}$, then $m = 0$. This is why the reference flux is also known as the "zero point" of the magnitude scale.

Magnitudes can be defined for different bands of wavelengths (see Sparke and Gallagher Chapter 1.1, especially 1.1.5), with different zero points for different color bands. What choice is conventionally made to set the $m = 0$ zero point?

(b) [4 points] Show that the difference in magnitudes of two stars is given by

$$m_2 - m_1 = 2.5 \log \frac{F_1}{F_2} \tag{2}$$

which does not depend on the zero point flux.

(c) [5 points] While the *apparent* magnitude $m$ of a star measures its flux, the *absolute* magnitude $M$ measures its luminosity. This is done by defining $M$ to be the apparent magnitude the star would have *if* it were at a distance $d_0 = 10$ pc (parsecs). If the true distance to the star is $d$, show that $M = m - 5\log(d/d_0)$.

We know the Sun's luminosity very accurately, so this forms a useful astronomical luminosity scale, similar to the way the Sun's mass is a useful unit of astronomical mass. Derive an expression for the ratio of a star's luminosity $L$ to the Sun's, i.e., $L/L_\odot$, in terms of star's absolute magnitude $M$ and the Sun's, $M_\odot$.

(d) [5 points] The limiting minimum brightness your eye can see (in the visible waveband $V$) is about $m_{\rm lim} = 6$ mag; this means that the eye can see fluxes having $m < m_{\rm lim}$. If a star has absolute magnitude $M$, give an expression for the distance $d_{\rm lim}$ (in parsecs) at which its apparent magnitude is $m_{\rm lim}$.

(e) [4 points] Then use Table 1.4 to find $d_{\rm lim}$ for the Sun, an A0 dwarf, and an M6 dwarf; use Table 1.5 to do the same for an M0 red giant, and use Table 1.6 to do the same for an A0 supergiant.

Comment on the implications for our view of the night sky. What biases might we have?

Finally, use Tables 1.4 through 1.6 to find the distance to the farthest single star the eye can see. What kind of star is this?

(f) [4 points] The light from a galaxy is due almost entirely to the light from its stars. A typical galaxy bright galaxy (called an $L_*$ galaxy) contains about $10^{11}$ stars. Assume these typically have $L = 0.2L_\odot$, and show that the absolute magnitude of a bright galaxy is about $M_* = -21$ mag. What is the maximum distance in Mpc (1 Mpc = 1 megaparsec = $10^6$ parsecs) at which an $L_*$ galaxy can be and still be visible to the eye? Compare this result to the typical galaxy-to-galaxy spacing of $\sim 5$ Mpc, and comment.

In fact, galaxies are not point sources but have a finite apparent angular size, i.e., their starlight is not concentrated in a point but rather spread over a patch of the sky. In view of this, is your point-source maximum distance estimate an underestimate or overestimate of the result for a real galaxy?