Astronomy 406, Fall 2013
Problem Set #10

Due in class: Friday, Nov. 15   Total Points: 60 + 5 bonus

1. *Cosmological FAQ.* The following are some frequently asked questions about cosmology (or some questions that should be asked more frequently, because they are great sources of confusion).

   (a) [5 points] Where did the big bang happen? Explain your answer in 1–2 sentences.

   (b) [5 points] Where is the center of the universe? Explain your answer in 1–2 sentences.

   (c) [5 points] What does it mean for the universe to have a "flat" geometry? Isn't space three-dimensional?

2. *The first law of thermodynamics and the cosmic equation of state*

   (a) [5 points] Pressure forces can do work if the volume changes. In class, we noted that the pressure force is $\vec{F} = \int P\, d\vec{A}$, where $d\vec{A}$ is an infinitesimal area element, pointed outward from (i.e., normal to) the surface, for example into a piston if there is one. If the pressure forces cause a change dV in volume (e.g., by moving a piston), then this does *work*. Given the usual definition of work $W = \int \vec{F} \cdot d\vec{x}$ Show that the work done is $W = \int P\, dV$. *Hint*: this isn't actually very hard. All you really need to realize is that when an area element is displaced by a distance $d\vec{x}$, then the volume swept out is just the area times the component of the displacement that is normal (perpendicular) to the surface.

   Then use this result to show that for an isolated system, conservation of energy implies that the work done is entirely at the expense of internal energy U, so that $dU = -PdV$. This result may be familiar to some as the *First Law of Thermodynamics*, and in fact assumes that the expansion or contraction is adiabatic, i.e., there is no heat going into or out of the system.

   (b) [5 **bonus** points] Show that the usual Friedmann equation for $\dot{a}$ and the Friedmann acceleration equation for $\ddot{a}$ together imply that the universe obeys

   $$d(\rho c^2 a^3) = -Pd(a^3) \tag{1}$$

   Hint: to see this, note that the usual Friedmann equation governs $\dot{a}$, while the Friedmann acceleration equation governs $\ddot{a}$. Take the time derivative of the usual Friedmann equation–in particular, find $d(\dot{a}^2)/dt$. The result will contain a term with $\ddot{a}$; then use the Friedmann acceleration equation to show that eq. (1) holds. Finally, interpret eq. (1) physically in light of your result from part (a).

   (c) [5 points] In general, the pressure P depends on ρ; the $P(\rho)$ relationship is known as the *equation of state*. As mentioned in class, the different components of the universe often obey $P = w\rho c^2$, where w is a dimensionless constant. Show that in this case, eq. (1) gives $\rho \propto a^{-3(1+w)}$.

   (d) [5 points] Note that for radiation, one can show that $P_{\rm rad} = 1/3\ \rho_{\rm rad} c^2$. Use this result and the previous one to show that this gives $\rho \propto a^{-4}$, which we had already derived by other means.

(e) [5 points] Show, on the other hand, that our result for the matter density $\rho_{\rm m} \propto a^{-3}$ demands that $w = 0$ and thus $P = 0$. This might seem odd: we know that normal matter *does* have pressure $P = nkT$ if it is at nonzero temperature. However, go on to show that $P \ll \rho c^2$ as long as $kT \ll mc^2$, which is exactly the condition that *defines* non-relativistic matter, and thus guarantees that we don't cheat much when we put $P \approx 0$.

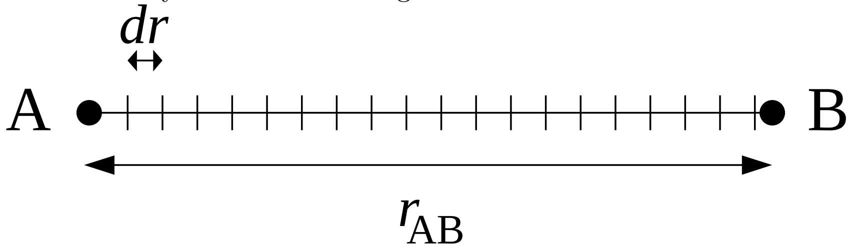
(f) [5 points] Show that the two Friedmann equations with a cosmological constant can be rewritten as the ordinary Friedmann equations if we define a *vacuum energy density* $\rho_\Lambda = \Lambda c^2/8\pi G$ and an associated vacuum pressure $P_\Lambda = -\rho_\Lambda c^2$. Use this vacuum energy equation of state with eq. (1) to show that $\rho_\Lambda \propto a^0 = const.$

(g) [5 points] Fits to the data from Type Ia supernovae have been done, assuming that the Universe contains both nonrelativistic matter, and another component (*dark energy*) characterized by some value of w. Analyzed this way, the data demand that the universe contains a large component of dark energy, which must have $w < -0.7$. Show that a universe that is accelerating (i.e., that has $\ddot{a} > 0$) requires that $w < -1/3$. Thus you will show that the supernova data demand that the universe is accelerating.

3. *Particle horizons.* Consider two observers (e.g., two galaxies A and B) separated by a fixed, *comoving* distance r_{AB} (which we can set to be their distance today). The true *physical* distance between the galaxies of course changes with time due to cosmic expansion; for each small comoving distance increment dr, the corresponding increment in physical distance is just given by $d\ell_{\rm phys} = a(t)dr$. Using this expression, we see that the entire physical distance is simply

$$\ell_{\rm phys}(t) = \int_{r_A}^{r_B} d\ell_{\rm phys} = a(t) \int_{r_A}^{r_B} dr = a(t) r_{AB} \tag{2}$$

as we already saw some time ago.



(a) [5 points] Now consider a light ray which moves from galaxy A to galaxy B. Its speed relative to the observers it passes is $v = d\ell_{\rm phys}/dt = a(t)dr/dt = c$. Using this, show that a light ray emitted at time $t = 0$, and detected at time t, travels a physical distance

$$\ell_{\rm hor}(t) = c\, a(t) \int_0^t \frac{d\tau}{a(\tau)} \tag{3}$$

This distance is known, for reasons you will soon clarify, as the *particle horizon.*

(b) [5 points] Compute the particle horizon size for both a matter dominated and a radiation dominated universe, using the known forms of $a(t)$ for each. You should find in both cases that $\ell_{\rm hor} \propto ct$; find the constant of proportionality in both cases.

(c) [5 points] Explain why a particle moving (with respect to the observers is passes) at speed $v < c$ will travel a physical distance $\ell(t) < \ell_{\rm hor}(t)$. But Special Relativity also demands that the speed of any passing particle is limited to $v < c$. Combine these facts to argue that $\ell_{\rm hor}(t)$ is the maximum distance, at any time t, which any particles can have traveled, and thus the maximum distance at which any information and/or signals can have moved, over the history of universe. In doing so, explain why the name "particle horizon" is appropriate for $\ell_{\rm hor}$.