Astronomy 406, Fall 2013
Problem Set #11

Due in class (or before!): Friday, Nov. 22 Total Points: 60 + 5 bonus

1. *Cosmology FAQ, Part Deux*

   (a) [5 points] Is it the so-called CMB, or the CMB properly so called?
   To answer this, dissect the name. In what way is the CMB "cosmic"? In what way is it "microwave"? In what way is it "background" radiation?

   (b) [5 points] How can we see the CMB if the photons were released 14 Gyr ago? Shouldn't those photons be billions of light years away by now?

2. *Properties of The Uniform Cosmic Microwave Background.* Today, the CMB has a temperature $T_0 = 2.725$ K. But while these photons are chilly today, they have important implications for the early universe.

   (a) [5 points] Cosmic recombination took place at redshift $z_{\rm rec} \approx 1100$.

      i. Find the temperature of the universe at recombination, expressing your answer in Kelvin.

      ii. Find the peak wavelength of CMB photons at recombination. To what part of the spectrum do these photons belong?

      iii. If you were present at recombination (in appropriate life support gear) what color would you see?

   (b) [5 **bonus** points] For a blackbody at temperature T, the number density of photons per frequency band $d\nu$ centered on frequency ν is

$$n_\nu = \frac{dn}{d\nu} = \frac{8\pi}{c^3} \frac{\nu^2}{e^{h\nu/kT} - 1} \tag{1}$$

   where, as usual, $h = 6.63 \times 10^{-27}$ erg s $= 4.14 \times 10^{-15}$ eV s is Planck's constant, and $k = 1.38 \times 10^{-16}$ erg K^{-1} $= 8.62 \times 10^{-5}$ eV K^{-1} is Boltzmann's constant.
   The energy density of photons per frequency band $d\nu$ centered on frequency ν is

$$\varepsilon_\nu = \frac{d\varepsilon}{d\nu} = h\nu \; n_\nu = \frac{8\pi \; h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{2}$$

   Show that the total number density of photons of all ν is

$$n = \int_0^\infty n_\nu \; d\nu = 16\pi \; \zeta(3) \left(\frac{kT}{hc}\right)^3 \tag{3}$$

   with $\zeta(3) = 1 + 1/2^3 + 1/3^3 + 1/4^3 = \sum_{n=1}^\infty 1/n^3 = 1.202$.
   Also show that the energy density of photons of all ν is

$$\varepsilon = \int_0^\infty \varepsilon_\nu \; d\nu = 48\pi \; \zeta(4) \frac{(kT)^4}{(hc)^3} = 7.57 \times 10^{-15} \text{ erg/cm}^3 \; T_{\rm K}^4 \tag{4}$$

   where $T_{\rm K}$ is the temperature in Kelvin.

(c) [5 points] Calculate the number density of CMB photons in the universe today, in photons/cm^3.

(d) [5 points] Find the contribution of the CMB to the cosmic density as follows.

   i. Calculate the CMB energy density today, in erg/cm^3.

   ii. Then find the equivalent CMB mass density ρ, in g/cm^3.

   iii. Finally, find $\Omega_{\rm CMB}$ today.

   iv. Comment on your result. Were we well-justified in ignoring radiation in our accounting of cosmic mass-energy today?

(e) [5 points] There is a redshift at which the CMB density $\rho_{\rm CMB}$ is equal to the matter density $\rho_{\rm matter}$. This is the "epoch of CMB-matter equality."

   i. Find this redshift. Note that today, $\Omega_{\rm matter} \approx 0.3$.

   ii. At redshifts higher than this, the universe is radiation dominated. Explain why.

   iii. Does recombination occur when the universe is matter dominated or radiation dominated?

3. *Fluctuations in the Cosmic Microwave Background.* Detection of fluctuations in the CMB was deemed worthy of a Nobel Prize in 2006. Here you will get a feeling for why this is a big deal.

   (a) [5 points] CMB temperature variations from one point in the sky to another are found to be very very small compared to the average temperature. Explain how this result provides good evidence for the isotropy of the universe.

   (b) [5 points] Consider the CMB temperature variation between two points on sky (directions 1 and 2):

$$\left(\frac{\delta T}{T_{\rm avg}}\right)_{\rm obs} = \frac{T_2 - T_1}{(T_2 + T_1)/2} \tag{5}$$

   as measured today, at $z = 0$. Both T_1 and T_2 have changed significantly since recombination, when $z_{\rm rec} \approx 1100$. Using the scale factor and thus redshift dependence of blackbody temperature to find expressions for these temperatures $T_{1,\rm rec}$ and $T_{2,\rm rec}$ at recombination. Then go on to show that $\delta T/T_{\rm avg}$ has the same value today as it did at recombination. Finally, explain why this result is crucial for our understanding of CMB.

   (c) [5 points] The detections of temperature fluctuation in the CMB marked a great triumph for cosmology, because it links the early universe to the formation of galaxies we see today. Explain the nature of this link.

4. *The cosmic neutrino background.* Neutrinos are produced thermally (as a neutrino "blackbody radiation") in the early universe, and linger today.

   (a) [5 points] In hot, dense early universe, neutrinos were able to interact with the matter in the universe, and were in thermal equilibrium with the rest of the cosmos. Consequently, neutrinos had a temperature, which (initially) was the same at that of photons. Moreover, since neutrinos were very relativistic, they were a kind of "blackbody radiation" themselves, so that the number densities were comparable to the usual blackbody photons: $n_\nu \approx n_\gamma$.

Since the early universe, both neutrinos and blackbody photons have been conserved. Again using the CMB photon number density, estimate the present number density of cosmic neutrinos (not a hard question!) and use it to estimate the total number of cosmic neutrinos in your body at any moment.

(b) [5 points] Neutrinos come in three types or "species," call ν_e, ν_μ, and ν_τ. Show that if a neutrino of species i is non-relativistic with mass $m_{\nu,i}$, then the total mass density in neutrinos is $\rho_{\nu,\mathrm{tot}} = (\sum_i m_{\nu,i}) n_\nu$, where $\sum_i m_{\nu,i}$ is the sum of the neutrino masses.

(c) [5 points] To date, neutrino masses are not yet measured (though we have solid evidence that the masses, whatever they may be, are nonzero). The best laboratory limits on neutrino masses suggest that $m_\nu c^2 < 1$ eV for all neutrino species. Use this result to now prove that $\Omega_\nu \ll \Omega_m$: neutrinos are not important as dark matter. Comment on the implications of this result.