## Astronomy 406, Fall 2013
## Problem Set #2

Due in class: Friday, Sept. 13;   Total Points: 60 + 5 bonus

1. *The Singular Isothermal Sphere.* The condition of hydrostatic equilibrium is an important one in all of astrophysics; in class we mentioned it in the context of stars, but it is more generally applicable.

   (a) [5 points] Hydrostatic equilibrium involves pressure forces. In general, these are related to the mass density ($\rho = dM/dV$) and temperature $T$ of the gas. In particular, systems are often well-approximated as *ideal gasses*. Then we have the relation $PV = NkT$, the ideal gas equation of state. Here $P$ is the pressure, $V$ is the volume, $N$ is the number of particles, and $k = 1.38 \times 10^{-23}$ Joule/K is Boltzmann's constant.

   Given the average mass $\mu$ of a gas particle, write the ideal gas equation of state in terms of $P$, $\mu$, $\rho$, and $T$. This form will be useful below and throughout the course.

   (b) [5 points] In class, we showed that a blob of gas in hydrostatic equilibrium has no net force on it: $-m_{\rm blob}g(r) - [P(r+h) - P(r)]A = 0$, where the blob has height $h$ and area $A$, and lies at radius $R$.

   Express this result in terms of the blob density $\rho(r)$, and show that as we let the blob height $h$ become small, we have

$$-\frac{dP}{dr} = \rho g \tag{1}$$

   which is the equation of hydrostatic equilibrium. Show also that for a spherically symmetric distribution of matter, $g(r) = Gm(r)/r^2$, where $m(r) = 4\pi \int_0^r R^2 \rho(R) dR$ is the mass enclosed inside radius $r$.

   (c) [5 points] Now combine the results from parts (a) and (b). If an ideal gas is in hydrostatic equilibrium, held at the same *constant* temperature $T$ throughout ("isothermal"), then show that it obeys

$$-\frac{r}{\rho}\frac{d\rho}{dr} = -\frac{d\ln\rho}{d\ln r} = \frac{G\mu m(r)}{kTr} \tag{2}$$

   (d) [5 points] The trick now is to solve eq. (2), i.e., to find $\rho(r)$ and thus $m(r)$ and hence the structure of gas in the star (or "gas" of stars in a galaxy). This is in general not trivial, but we can take the optimistic approach and try a simple solution. Consider a power-law density structure: $\rho(r) = Cr^{-b}$, where $C$ and $b$ are constants. Show that this form can satisfy eq. (2), but only if $b$ and $C$ take particular values. Find these values. *Hint:* when you use the power law $\rho(r)$ and the accompanying $m(r)$ in eq. (2), you will find a relationship which depends on $r$. Recall that this relationship must hold for *any* value of $r$–this will force a choice of $b$ and then $C$.

   (e) [5 points] Using your results from (d), find $m(r)$. Use this compute the speed $v_c(r)$ of a test particle in a circular orbit at radius $r$. *Hint:* recall your answer to question 1 of Problem Set 1. How does $v_c(r)$ depend on $r$?

(f) [5 points] Comment on the behavior of $\rho(r)$ and $M(r)$ as $r \to 0$ and $r \to \infty$. What does this imply for our solution? If all has gone well, you will see why this is known as the *singular* isothermal sphere.

2. *Trigonometric Parallax Redux.* The usual discussion of trigonometric parallax places the star whose distance we measure in the *same plane* as the Earth's orbit around the Sun (i.e., the ecliptic plane). But of course this is generally not the case.

   (a) [5 points] First, consider a nearby star that *does* lie in the ecliptic plane (declination $\delta = 0$), as usually shown in the diagram. Over the course of a year, how will the star move with respect to the celestial sphere (i.e., with respect to very distant objects that show no parallax). What will the path look like on the sky? If we know the star's right ascension $\alpha$, when should the parallax observations be taken?

   (b) [5 points] Now consider a star that lies at the north celestial pole (declination $\delta = +90^\circ = 90^\circ$ N). Over the course of a year, how will the star move with respect to the celestial sphere–what will the path look like on the sky? Explain how measurements of the star's path can provide the parallax angle $p$ and thus the stellar distance. *Hint:* a sketch will be very useful here.

   (c) [5 **bonus** points] For the trifecta, now consider a star with intermediate declination $0^\circ < \delta < +90^\circ$. What is the annual path of such a star on the celestial sphere? Explain how measurements of the star's path can provide the parallax angle $p$ and thus the stellar distance. *Hint:* a sketch will be extremely useful here.

3. *Collision Technology and the Interstellar Obscuration.* The scattering of photons and other particles as they propagate through a medium is a situation we will frequently encounter, and so it will be very useful to develop some technology to understand this.

   Consider a particle (the "projectile") which we will denote $a$, which travels through a medium filled with particles of type $b$ (the "targets"). Then *number density*, i.e., number of particles per unit volume, of particle $b$ is $n_b$. We wish to understand the chances of collisions of $a+b$; this problem is very similar to that throwing darts at a dartboard, with the targets as bullseyes. Let $\sigma$ the bullseye "size"–really, the cross-sectional area–that the targets present. Finally, the speed of the projectiles ia $v$.

   (a) [5 points] We wish to find the probability $P$ that a projectile will hit a target. To do this, imagine that a box or slab of area $A$ and thickness $\Delta x$, filled with targets of number density $n_b$. This is the dartboard. Consider a single particle of type $a$, fired into the box in the $x$-direction, but in a random location in $y$ and $z$. This corresponds to a single "dart."

   First, find the number $N_b$ of targets–and thus bullseyes–in the box (not a hard problem!). Assume that the target particles don't overlap with each other, and explain why the probability of a "hit" by the "dart" $a$ is $P = N_b\sigma/A$, which in turn means that

   $$P_{\text{collision}} = n_b \ \sigma \ \Delta x \tag{3}$$

   Go on to explain why the formula makes sense: if the target density is increased but $\Delta x$ held fixed, what happens? If $n_b$ is fixed but $\Delta x$ decreased, what happens? Also, if $n_b$ and $\Delta x$ are held fixed, what happens if we change the dartboard size $A$?

(b) [5 points] For a fixed target density $n_b$, find how thick we should make our slab to make $P_{\text{collision}} = 1$? This length is known as the *mean free path* $\lambda_{\text{mfp}}$; explain why this name is appropriate. What do you expect happens if $\Delta x > \lambda_{\text{mfp}}$?

(c) [5 points] If the projectiles move with speed $v$, find how long it takes them to move a distance $\lambda_{\text{mfp}}$? This is known as the *mean free time* $\tau$. Explain why the name is appropriate.

(d) [5 points] Now we are in a position to understand a bit more about dust obscuration in our Galaxy (and the Earth!). In the interstellar medium, the hydrogen gas number density varies widely, but on average is about $n_{\text{H,avg}} \sim 1$ atom $\text{cm}^{-3}$. For about every $10^{12}$ H atoms, there is one dust particle, of average size about $r \simeq 0.3\ \mu\text{m} = 3 \times 10^{-5}$ cm. Find $n_{\text{dust}}$, and find $\sigma$ using the geometric size of the dust. Use these to find $\lambda_{\text{mfp}}$ for photons in the interstellar medium. Compare with the size of our Galaxy and the distance to its center, and comment.

Finally, the atmosphere on Earth's surface is about $10^{20}$ (!) times denser than the average interstellar medium. If our atmosphere contained the same proportion of dust as is found in interstellar space (that is, the same ratio of dust particles to gas atoms), find $\lambda_{\text{mfp}}$. Comment on the result.