Astronomy 406, Fall 2013
Problem Set #6

Due in class: Friday, Oct. 11; Total Points: 60 + 5 bonus

1. *Stellar Orbits in a Uniform Density Cluster.* Consider a spherical star cluster with a uniform, constant mass density ρ_0.

   (a) [5 points] The gravitational acceleration $\vec{g}(r)$ varies with radius.

   i. Find $\vec{g}(r)$ at all distances r inside the sphere.

   ii. Show that, inside the sphere, we can write $\vec{g}(r) = -\omega^2\,\vec{r}$, where ω is the inverse of a gravitational timescale, and find an expression for ω.

   (b) [5 points] In class we showed that, in a gravity field of this form, any star's orbit is confined to a plane. Within that plane, we may choose Cartesian coordinates $\vec{r} = (x, y)$ centered on the cluster center.

   i. Solve the equation of motion for a star that orbits inside this cluster (i.e., does not move outside). Show that the solution may be written

$$x(t) = a\ \cos(\omega t + \alpha) \quad (1)$$

$$y(t) = b\ \cos(\omega t + \beta) \quad (2)$$

   where a, b, α, β are constants.

   ii. Show that the angular momentum per unit mass $\vec{\ell} = \vec{L}/m$ has magnitude $\ell = \omega ab\,|\sin(\alpha - \beta)|$.

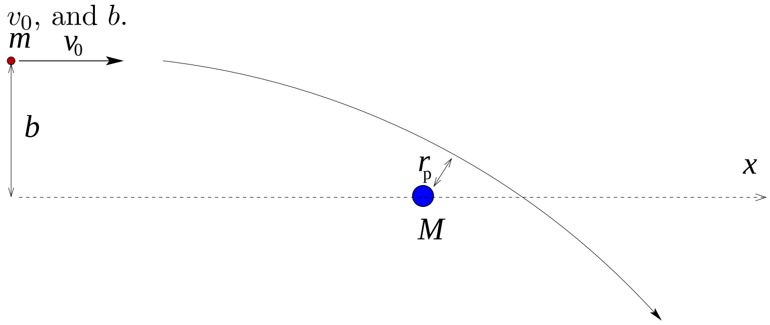
   (c) [5 points] This orbit is an ellipse centered on the cluster center.

   i. Show this by finding the relation between α and β when the orbit is (a) a line, (b) an ellipse with major axis a and minor axis b.

   ii. Show that the orbit period is twice the time ΔT_r to go from one orbit radial extremum to the next.

2. *Gravitational Billiards: Stellar Scattering.*

   (a) [5 **bonus** points] Consider a particle of mass m in an unbound ($E_{\rm tot} > 0$, hyperbolic) trajectory past a point mass M at the origin. As shown in the figure, the particle initially has speed v_0, is very distant along the x-axis ($|x| \to \infty$), and is at $y = b$. Due to the gravity of M, the particles' path will be bent, an it will come within some closest radial distance ("pericenter") r_p. We wish to find r_p in terms of M, v_0, and b.



To do this, first note that through conservation of energy, you can find an expression relating the *magnitude* of the velocity $v(r)$ at any radius r, in terms of r, M, and v_0.

Then consider angular momentum. Explain why the initial magnitude of the angular momentum is $L_{\rm init} = mv_0 b$. Then explain why for most finite r, the magnitude $L \neq mvr$, but that at pericenter, we do have $L_p = mv_p r_p$, where $v_p = v(r_p)$ is the speed at pericenter. Use conservation of angular momentum to then relate $v(r_p)$ to r, b, and v_0

Finally, combine the conservation of energy result and the conservation of angular momentum result to eliminate v. This should give a quadratic equation whose appropriate solution is

$$r_{\rm closest} = r_p = \sqrt{b^2 + r_{\rm grav}^2} - r_{\rm grav} \qquad (3)$$

where for convenience we have put $r_{\rm grav} = GM/v_0^2$.

(b) [5 points] In eq. (3), the lengthscale $r_{\rm grav} = GM/v_0^2$ measures a characteristic gravitational "zone of influence" for scattering (very much like the Einstein radius does for lensing).

i. To explore why this is, first consider the case $M = 0$. What should the particle motion be then? In this case, find $r_{\rm closest}$, and show that it makes sense.

ii. Now consider the opposite limit, where $r_{\rm grav} \gg b$. Show that in this case, $r_{\rm closest} \ll b$, and thus the scattering should be significant.

(c) [5 points] Consider the case of scattering off a star of mass $M = M_\odot$ with a relative speed $v_0 \approx 3$ km/s typical of solar neighborhood stars.

i. Find $r_{\rm grav}$ and express your answer in pc.

ii. If b is a typical spacing between stars, show that $r_{\rm grav} \ll b$, and comment on the significance of this result.

(d) [5 points] The cross section for gravitational scattering is roughly $\sigma \approx \pi r_{\rm grav}^2$.

i. Use this, and an estimate of the local number density of stars as found in PS 2, to find the mean free path and mean free time for star-star scattering in our Galaxy.

ii. Compare these with the size and age of the Milky Way, and comment.

(e) [5 points] Finally, galaxies can and do collide with each other, with relative speeds v which can be 100 km/s or more.

i. Comment on the chances of star-star scatterings and collisions during these encounters.

ii. On the basis of your answer, how would you expect stars to behave in galaxy collisions? How about interstellar gas–would you expect this to behave differently?

3. *Globular Clusters and the Virial Theorem.*

(a) [5 points] Consider a globular cluster of typical mass and tidal radius.

i. Estimate the root-mean-square virial speed $v_{\rm rms}$ for such a cluster.

ii. Compare this speed to the observed line-of-sight (radial) values v_r in SG Table 3.1, and comment. To do this, note that, in terms of the vector components in spherical coordinates, $v^2 = v_r^2 + v_\theta^2 + v_\phi^2$. On average, each of these components has the same magnitude, so that so $v_r^2 = v^2/3$.

(b) [5 points] For a globular cluster system, show that the root-mean-square speed $v_{\rm rms}$ and the escape speed $v_{\rm esc}$ are related by $v_{\rm rms} \simeq v_{\rm esc}/2$. Hint: you will want to find the total potential energy of the cluster, and use the Virial theorem.

4. *Surface Brightness and Projection Effects.*

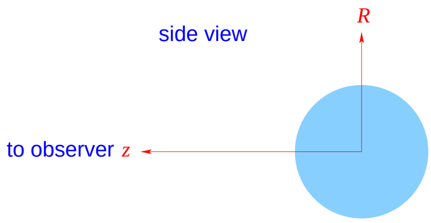
(a) [5 points] Surface brightness is independent of distance, in the absence of absorption and cosmological effects.

i. Explain clearly why surface brightness is independent of distance, and what this means physically for the appearance of Galaxies.

ii. Stars are also extended objects and thus have their own surface brightness. Yet we typically use the inverse square law to describe a *decrease* of stellar flux with distance. Why isn't star surface brightness constant and thus the flux independent of distance?

(b) [5 points] *Corrected version!* Consider a spherical distribution of stars with the number density of stars $n(r) = K\ r^{-s}$ falling off is an inverse power law of distance r from the center, trailing off indefinitely to infinity.

When we observe this object on the sky, it appears flattened along the line of sight, and thus the brightness pattern we see is the *projected* density along the sightline. We can describe this by using cylindrical coordinates (R, z), centered on the center of the star cluster, as shown in the diagram at right. Let z lie along the line of sight, and R lie tangential to the line of sight, i.e., in the plane of the sky. Note that $r^2 = R^2 + z^2$, and the cluster interior (at least formally) spans $z \in [-\infty, +\infty]$.



Calculate the projected surface density $\Sigma(R) = \int_{-\infty}^{\infty} n(R, z)\ dz$. Show that $\Sigma(R) \propto R^{-(s-1)}$.

(c) [5 points] In the webpage for the Oct. 2 class meeting, there is a link to the surface density profile for the globular cluster NGC 6338.

i. What is plotted is $\log \Sigma$ versus $\log r$. In the outer regions of the cluster, the results are well fit by a linear relation $\log \Sigma = -\beta\ \log r + const$. Estimate the value for β, and use this to find the power-law index s for this cluster

ii. Comment on the nature of the cluster density profile.