Astronomy 406, Fall 2013
Problem Set #9

Due in class: Friday, Nov. 8 Total Points: 60 + 5 bonus

1. *The Care and Feeding of Supermassive Black Holes.* The most distant known QSO has a redshift of $z = 7.085$, and a luminosity $L = 6.3 \times 10^{13} L_{\odot}$ (!). Here we investigate the growth history of this amazing object, known as ULASJ1120+0641. For the purposes of this problem, we will assume this object has a black hole in its nucleus, which is accreting matter and radiating isotropically.

   (a) [5 bonus points] Recall that photons of energy E have momentum given by $cp = E$. Using this, show that the rate of radiation momentum flow at radius r is $d\mathcal{P}_{\rm rad}/dtdA = F/c = L/4\pi cr^2$.

   Then consider ionized hydrogen gas at a distance r from the black hole. The radiation scatters off both the protons and the electrons, but the scattering off protons is negligible. The photon scattering on electrons is called Thomson scattering, and acts as if the photons "see" the electrons with an area $\sigma_{\rm T} = 6.6 \times 10^{-25}$ cm^2, known as the Thomson cross section. Using this, and the radiation pressure, find the radiation force on an electron. Comment on how this force depends on distance.

   Show that the radiation force on the electron and proton exactly balances the gravitational force when the luminosity is

   $$L_{\rm Edd} \approx \frac{4\pi GMm_pc}{\sigma_{\rm T}} \tag{1}$$

   where we sued the fact that $m_p \gg m_e$ and thus $m_p + m_e \approx m_p$. This is known as the *Eddington limit* to luminosity.

   Finally, explain why this is the maximum luminosity a mass M can have, by considering what would happen if $L > L_{\rm Edd}$.

   (b) [5 points] Assume that ULASJ1120+0641 radiates at the Eddington limit given by eq. (1).

      i. Calculate the mass of the black hole, expressing your answer in units of $M_{\odot}$.

      ii. Compare your result with the mass of Sgr A*, and comment.

   (c) [5 points] When matter of mass $m_{\rm acc}$ is accreted onto a black hole, the maximum energy that can be radiated is $E = \varepsilon m_{\rm acc}c^2$, where the "efficiency" $\varepsilon \approx 0.1$. Show that the luminosity is thus related to the accretion rate by $L = \varepsilon \dot{M} c^2$. Here $\dot{M}$ is the accretion rate, i.e., the rate at which mass falls on the the black hole.

   (d) [5 points] Use the results from parts (b) and (c) to show that an accreting black hole grows with time according to

   $$\dot{M} = \frac{M}{\tau_{\rm acc}} \tag{2}$$

   with $\tau_{\rm acc} = \varepsilon\sigma_{\rm T}c/4\pi Gm_p = 5 \ \times 10^7$ years.

   (e) [5 points] Finally, suppose that ULASJ1120+0641 has grown by accretion for the entire age of the universe until $z = 7.085$; this corresponds to about $t_{\rm obs} = 8 \times 10^8$ yr = 800 million years.

   i. Solve eq. (2) for $M(t)$. and find the initial mass of the black hole.
   ii. Is its possible to grow a black hole of the observed mass starting from a stellar mass black hole of $M \sim 10-100 M_\odot$ created in a very early supernova explosion?

2. *A Static Universe.* As mentioned in class, Einstein initially invented a new constant of nature, the cosmological constant Λ, which amends the usual rules of gravitation (that is, General Relativity) so that now the Friedmann equations read

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3} \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{4}$$

   His original goal in doing this was to keep the universe static and non-expanding.

   (a) [5 points] In this picture, the density ρ_0 of the universe is now constant in space *and* time. Also you may assume $P = 0$.
      i. Find the critical value Λ_c of the cosmological constant which makes the universe static, so that $\dot{a} = \ddot{a} = 0$. This solution is called the *Einstein static universe*.
      ii. Also show that in this model, we must have $k = +1$, and give an expression for the curvature radius R_0.

   (b) [5 points] Consider an Einstein static universe in which the density is now perturbed, so that $\rho(\vec{r})$ fluctuates from point to point in space around its average value ρ_0. Discuss (but don't calculate) what would happen to a region in which $\rho(\vec{r}) > \rho_0$ and one in which $\rho(\vec{r}) < \rho_0$. Comment on the implications for the Einstein static universe.

3. [5 points] *Escape Speed and* Ω. Using the Newtonian picture we developed in class, find the escape speed $v_{\rm esc}$ for an arbitrary cosmological *comoving* sphere which always encloses mass M. Express your result in terms of the sphere's density $\rho(t)$ and radius $R(t)$. Then use Hubble's law to find the speed v with which the sphere expands. Finally, show that the condition $v \geq v_{\rm esc}$ is equivalent to $\Omega \leq 1$, and $v < v_{\rm esc}$ gives $\Omega > 1$. Interpret your result physically.

4. *The Friedmann Equation: Limiting Cases.* In class, we studied a *matter-dominated* universe, and explicitly solved for some properties of this universe. Here we will extend this treatment to also include a radiation-dominated universe, in which $\rho = \rho_{\rm rad}$, and a curvature-dominated universe in which $c^2/R_0^2 a^2 \gg 8\pi G\rho/3$.

   (a) [5 points] Show that in a radiation-dominated universe, $a(t) = (t/t_0)^{1/2}$, and in a curvature-dominated universe (with $k = -1$) that $a(t) = t/t_0$.

   (b) [5 points] Find $H(t)$ and $z(t)$ for a radiation-dominated universe and for a curvature-dominated universe. For each case, what is the age of the universe at $z = 1$, expressed in terms of t_0? What is the redshift at which the universe is 10% of its present age?

(c) [5 points] Using the $a(t)$ solutions above, find $\rho(t)$ the case of a radiation-dominated and then in the case of a matter-dominated universe. In both cases, express your answer only in terms of t and physical constants like G (but not including H_0 and ρ_0). Also show the relationship between ρ_0 and t_0 for each case. *Hint:* consider the Friedmann equation as an expression for ρ.

5. *The Age of the Universe.*

   (a) [5 points] Show that for each of the cases of a matter-dominated, radiation-dominated, and curvature-dominated universe, the present age t_0 of the universe is related to the present value H_0 of the Hubble parameter by $t_0 = \beta/H_0$, where β is a constant which depends on which type of universe we are considering. Find the value of β for each case. Your calculations give the *expansion age* of the Universe.

   (b) [5 points] One of the best current methods of directly measuring the age of the universe comes from the ages of globular clusters. In class, we saw that analysis of globular clusters gives the limit $t_0 > 10.4$ Gyr. Using this and current estimates of the Hubble constant, place a limit on the observed value of β. Compare this to the models considered in part (a), and comment on your result.