Astro 406

Lecture 30

Nov. 6, 2013

Announcements:

- **PS 9 due Friday**
- Planetarium makeup due Wednesday

Last time: began cosmic dynamics

physical intuition: Pop fly analogy

Q: why and how is the Universe like a pop fly?

a matter-only, pressureless universe, $K = 0$ universe

Q: why is this not crazy?

Q: $a(t)$*? fate?* $H(t)$*?*

a matter-only, pressureless universe has **acceleration**

$$\ddot{a} = -\frac{4\pi}{3} G\rho_0 \, \frac{1}{a^2} \tag{1}$$

formally identical to $\ddot{r} = -GM_{\rm earth}/r^2$

$\rightarrow$ motion of a *pop fly*: ball launched vertically

in a *matter-only, pressureless universe* with $K = 0$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \tag{2}$$

- fate: in the far future, as $t \rightarrow \infty$
  $a \rightarrow \infty$! *expands forever!*
- expansion rate $H(t) = 2/3t$: not constant
  $H \rightarrow 0$ as $t \rightarrow \infty$

Carding the Universe: Does the Age Check Out?

In any cosmo model: H_0 and t_0 related
but we can measure *both* values *independently*
$\rightarrow$ see if connection holds up

Since we know

$$H_0 = 72 \text{ km sec}^{-1} \text{ Mpc}^{-1} \tag{3}$$

we can solve for the "Hubble time"

$$t_{\text{Hub}} = 13.8 \text{ billion years} = 13.8 \text{ Gyr} \tag{4}$$

for matter-only universe: expansion age $t_0 = \frac{2}{3} t_{\text{Hub}} = 9.1 \text{ Gyr}$

Oldest star (globular clusters) ages: $t_\star \sim 12 - 14$ Gyr, and $t_\star >$ 11.2 Gyr
thus we observe: $H_0 t_0 > 0.81$ and thus $\neq 2/3$
Q: what's going on?

Cosmic Fail: What is to be Done?

we found: matter-only, pressureless, $K = 0$ universe
does not agree with observations of cosmic age
$\rightarrow$ *this model ruled out!* not our universe!
$\rightarrow$ need to change the model

recall: derivation was (quasi)-Newtonian, non-relativistic
but we know the correct description must include
- special relativity, and
- general relativity

fortunately, much stays the same
but we get a deeper understanding

Cosmic Density Evolution: Radiation

radiation by definition: relativistic particles
$\rightarrow v \sim c, \; kT \gg mc^2$

Q: examples? what particles are relativistic today?

if not emitted/absorbed
Q: how does number density n_{rad} depend on a?
Q: how does single-particle E_{tot} depend on a–think photons?
Q: how does energy density ε_{rad} depend on a?

radiation (relativistic $\rightarrow$ $v \sim c$, $kT \gg mc^2$)
example–photons: $m_\gamma = 0$ and $v = c \Rightarrow$ always relativistic
electrons: $m_e \neq 0 \Rightarrow$ relativistic at $T \gtrsim m_e c^2/k \sim 10^{10}$ K
neutrinos: $m_\nu \neq 0$, but small $\Rightarrow$ maybe relativistic today?
certainly relativistic in the hot early universe

number density: if not created/destroyed (i.e., conserved)
then same "volume dilution" result holds:
$n_{\rm rad} \propto a^{-3}$

energy density $\varepsilon_\gamma = \langle E_\gamma \rangle n_\gamma$
ex: photons have $m_\gamma = 0$, but $E_\gamma \neq 0$
but $\langle E_\gamma \rangle \propto 1/a$
$\rightarrow$ $\varepsilon_\gamma \propto a^{-4}$ and so $\varepsilon_\gamma \propto T^4$ if thermal: Planck result!

$$\rho_{\rm rad} = \varepsilon_{\rm rad}/c^2 \propto a^{-4}$$

pressure EM theory says $P_\gamma = \varepsilon_\gamma/3$
holds generally for radiation: $P_{\rm rad} = \varepsilon_{\rm rad}/3$

The Friedmann Equation Revisited

Einstein 1917: first cosmological solutions in General Relativity
but for *non-expanding*, static universe (PS9)
thanks to "fudge factor" Λ

first solution for expanding universe: Alexander Friedmann (1924)
ignored Λ, but otherwise very general, allowing for:

- matter and energy density of any kind
  total cosmic mass density $\rho = \varepsilon_{\rm tot}/c^2$
  where $\varepsilon_{\rm tot} = \varepsilon_{\rm matter} + \varepsilon_{\rm rad} + \ldots$
- pressure of any kind
  $P_{\rm tot} = P_{\rm matter} + P_{\rm rad} + \ldots$

Friedmann acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G \left(\rho + \frac{3P}{c^2} \right) \qquad (5)$$

iClicker Poll: Relativistic Cosmodyanmics

Friedmann acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\left(\rho + \frac{3P}{c^2}\right) \tag{6}$$

consider universes with both *matter* and *radiation*

Based on Friedmann, wow will such universes evolve with time?

A always decelerate, may or may not recollapse

B always decelerate, must recollapse

C always accelerate, may or may not recollapse

D always accelerate, must expand forever

E none of the above

Friedmann says:

$$\ddot{a}a = -\frac{4\pi}{3}G\left(\rho + \frac{3P}{c^2}\right) \tag{7}$$

but $\rho > 0$ always: no negative mass!
and both matter and radiation have $P \geq 0$

thus $\rho + 3P/c^2 > 0$ for arbitrary matter and radiation
and thus $\ddot{a} < 0$: these universes *decelerate*

but pop fly intuition tells us:
final fate not determined by this equation alone
depends on gravity vs inertia $\rightarrow$ total energy

The Friedmann Equation Revisited

Friedmann equation for relativistic cosmic "energy" :

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{\kappa c^2}{R_0^2 a^2}$$

- "energy" equation stays formally the same!
- P does not appear!

"total energy" constant now has become

$$K = \frac{\kappa c^2}{R_0^2} \quad (8)$$

where

- c is speed of light
- R_0 a *lengthscale*
- $\kappa = \pm 1, 0$ records *sign of "total energy"*

Relativistic Cosmology

So far: quick-n-dirty Newtonian derivation
grafted scale factor onto non-relativistic gravity
got the right basic answer, but *ad hoc* approach

Correct, self-consistent solution:
relativistic gravity, i.e., **General Relativity**
⋆ dynamic spacetime a natural, built-in feature
⋆ curved spacetime automatically built-in
⋆ scale factor emerges automatically
⋆ redshifts, time dilation, other effects automatic
⋆ can naturally, correctly include relativistic matter, $P \neq 0$

Now can interpret parameters:

- R_0 lengthscale: **spatial curvature of Universe**
- $\kappa = +1$, 0, or -1: encodes **cosmic geometry**

Geometry of the Universe

cosmic *geometry* ("*curvature*") $\Leftrightarrow$ Friedmann eq. κ
3 possible spatial geometries $\Leftrightarrow$ three values for κ
choice fixed once and for all
$\rightarrow$ experimental question: which describes our universe?

$\boxed{\kappa = 0}$ no curvature: space Euclidean or "**flat**"
Δ angle sum $= 180°$, circle $A = \pi r^2$
total spatial volume $= \infty$

$\boxed{\kappa = +1}$ positive curvature: space is "**sphere-like**"
triangle angles sum $> 180°$, circle $A < \pi r^2$
volume finite

$\boxed{\kappa = -1}$ negative curvature; "like a saddle" or **hyperbolic**
Δ angle sum $< 180°$, circle $A > \pi r^2$
volume $= \infty$

Dr. Friedmann's Amazing Equation

fundamental equation of Cosmology!

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{\kappa c^2}{R_0^2 a^2} \tag{9}$$

predicts cosmic past and future!

Output:

- solving Friedmann gives $a(t)$ for all time

Input:

- Friedmann calls for particular properties of universe

Q: Namely, what info needed to solve for $a(t)$?

Solving Friedmann

Friedmann sez

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{\kappa c^2}{R_0^2 a^2} \tag{10}$$

i.e., evolution of the Universe (i.e., of a)
depends on what's in the universe

Namely, need to know:

1. cosmic mass (mass-energy) density ρ
   and $\rho(a)$ dependence

2. Need cosmic parameters (fixed numbers)
R_0: curvature scale of the U.
$\kappa = \pm 1, 0$: cosmic geometry

Cosmic Epochs

Cosmic density ρ is sum of all cosmic constituents
- matter is present for sure (both dark, baryonic)
- radiation present for sure *Q: in what form(s)?*

So at minimum, realistic universe has

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}(\rho_{\mathrm{m},0} a^{-3} + \rho_{\mathrm{rad},0} a^{-4}) - \frac{\kappa c^2}{R_0^2} a^{-2} \tag{11}$$

Q: which term most important at early times?
Q: which term most important at late times?
Q: sequence of events?

diagram: $\log H^2$ *vs* $\log a$

$\Rightarrow$ 3 major epochs

Cosmic choreography: sequence fixed

- **Radiation dominated**: early U $\rightarrow$ small a
  $\rightarrow \rho_{\text{rad}} \propto a^{-4}$ largest term in H

- **Matter dominated**: $\rho_{\text{matter}} \propto a^{-3}$
  drops less rapidly than radiation $\rightarrow$ dominates eventually

- **Curvature dominated** (if $\kappa \neq 0$): a^{-2} smallest dropoff
  $\rightarrow$ dominates at late times

Rewrite Friedmann:

at any time

$$\kappa \frac{c^2}{R_0^2 a^2 H^2} = \frac{\rho}{3H^2/8\pi G} - 1 \tag{12}$$

and so for *today*:

$$\kappa \frac{c^2}{R_0^2 H_0^2} = \frac{\rho_0}{3H_0^2/8\pi G} - 1 \tag{13}$$

So what?

Q: what if right-hand-side = 0? < 0? > 0?

Q: in practice, what is good about right-hand-side?

Q: what measurable number(s) do we need to know?

Rewrite:

$$\kappa \frac{c^2}{R_0^2 a^2 H^2} = \frac{\rho}{3H^2/8\pi G} - 1 \equiv \Omega - 1 \tag{14}$$

Define **critical density**

$$\rho_{\rm crit} = \frac{3H^2}{8\pi G} \overset{\rm today}{=} \frac{3H_0^2}{8\pi G} \tag{15}$$

Define: dimensionless **density parameter**

$$\Omega = \frac{\rho}{\rho_{\rm crit}} \tag{16}$$

★ since we know $\kappa = +1, -1$, or 0, then

$$\begin{array}{lclcl} \Omega > 1 & & +1 & & \text{spherical} \\ \Omega < 1 & \rightarrow \quad \kappa = & -1 & \rightarrow \quad \text{geometry is} & \text{hyperbolic} \\ \Omega = 1 & & 0 & & \text{Euclidean} \end{array} \tag{17}$$

★ $\Omega \Leftrightarrow \kappa \Leftrightarrow$ cosmic geometry!

Midterm Exam

Not for the fainthearted

I was generally pleased with how people rose to the challenge

average: 106.7 out of 120
median: 111 out of 120
standard deviation: 14

recall:
exam is worth the same as 2 Problem Sets

- if you're pleased, congrats! – but still work hard on the PS
- if you had a bad day, don't let it go to your head,
  work hard on PS