Astronomy 406, Fall 2013
Problem Set #7

Due in class: Friday, Oct. 25; Total Points: 60 + 7 bonus

1. *Spiral Rotation Curves.*

   (a) [5 points] SG Figure 5.4 gives the I-band (in the infrared) surface brightness $I(R)$ of NGC 7331 as a function of *angular* radius R, and SG eq. 5.1 gives a simple fitting formula for the profile. Use this to show that the total disk flux is $F = 2\pi I(0) h_R^2$.

   (b) [5 free points] From the surface brightness appeaing in Figure 5.4, one can infer the I-band luminosity. However, to do this accurately involves a more tedious and detailed calculation than I intended, and so you may skip this calculation and just use the result $L_I \approx 3 \times 10^{10} L_{I,\odot}$.

   (c) [5 points] Assuming that NGC 7331 stars have solar colors, use the I-band luminosity to compute the visible luminosity L_V, expressed in $L_{V,\odot}$. How does this compare to the Milky Way?

   (d) [5 points] A rotation curve for NGC 7331 appears in SG Figure 5.20.

      i. Use this to calculate the total mass in this galaxy, at the farthest radius the measurements allow you to go, $R = 37$ kpc; express your answer in $M_\odot$.

      ii. Find the V-band mass-to-light ratio of NGC 7331, compare this with the local solar neighborhood of $M/L_V \approx 2M_\odot/L_\odot$, and comment on the implications.

2. *Local Group Timing.* The Milky Way and M31 form a gravitationally bound system that dominates the Local Group. The MW-M31 dynamics are interesting in their own right, and also provide a way of obtaining a dynamical estimate of the total mass of the two Galaxies. The calculation is a bit long but is straightforward, and is a good workout for your astrophysical muscles, which will pay dividends later in the course.

   (a) [5 **bonus** points]. Consider the motion of two masses in a bound orbit around their common center of mass, with no angular momentum. The particles begin together, then move radially outwards, then fall back. Thus the equation of motion of the particles (really, the distance between them) is

   $$\ddot{r} = -\frac{GM}{r^2} \tag{1}$$

   where $M = M_1 + M_2$ is the total mass of the particles.

   Show that the solution to this equation, $r(t)$, can be written in the following form (compare SG eq. 4.24, for an orbit with semimajor axis a and eccentricity $e = 1$):

   $$\begin{aligned} r(\eta) &= a(1 - \cos\eta) && (2) \\ t(\eta) &= T(\eta - \sin\eta) && (3) \end{aligned}$$

   where a and T are constants.

   Note that we don't directly write $r(t)$, but instead express both r and t in terms of a third parameter η. This is known as a "parametric solution" to the equation

(1), and the dimensionless η is sometimes called the "development angle." As η takes different values, r and t change in such a way that r versus t is the particle's motion.

(b) [5 points] Now let's get a feel for how eqs. (2) and (3) work (whether you derived them or are willing to take them as a generous inheritance from Newton). The trick is find the appropriate values of η that correspond to interesting points in the particles' orbits.

   i. First: show that $t = 0$ corresponds to $\eta = 0$, and that $r = 0$ at this instant, as we required in the initial conditions.

   ii. Then show that the maximum separation between the particles occurs at $\eta = \pi$, and find the particle separation at this instant, as well as the time at which this occurs, expressing both in terms of a and T.

   iii. Finally, show that the particles come back together at $\eta = 2\pi$, and find the time at which this occurs. This is the period P of the motion

(c) [5 points] As noted above, this problem is an instance the general Keplerian elliptical motion of two point particles under gravity, for the special case of eccentricity $e = 1$. For such a motion, the maximum and minimum separations of the bodies are given by $(1 \pm e)a$. Show that this formula holds in our case, and that we can identify a in eq. (2) with the semi-major axis.

(d) [5 points] Because Kepler's laws apply for this problem, we can use Kepler's third law to relate the period P (and thus T) and semi-major axis a to the total mass M.

   i. Use this relationship to solve for M in terms of a and P.

   ii. Our job is now to find a and P in terms of observable quantities, so that we can solve for the total mass M.

   Show that we can write the relative velocity as

$$\frac{dr}{dt} = \frac{a}{T}\frac{\sin\eta}{1-\cos\eta} = \frac{r(\eta)}{t(\eta)}\frac{\sin\eta(\eta - \sin\eta)}{(1-\cos\eta)^2} \tag{4}$$

   and verify that dr/dt goes to the appropriate value at the time of maximum separation.

(e) [5 points] Now we will apply our solution to the case of the Milky Way – M31 system. We will assume that the two galaxies started off together (at the big bang, $t = 0$), and have since moved according to eqs. (2) and (3). Denoting the present epoch with subscript 0, we have $r(\eta_0) = r_0 = 770$ kpc, and $t(\eta_0) = t_0 = 13.7$ Gyr. We are now approaching M31, so we know that today, $(dr/dt)_0 = -120$ km/s.

   Using eq. (4), show that $dr/dt < 0$ implies that $\pi < \eta_0 < 2\pi$.

(f) [5 points] Using the present values of r, t, and dr/dt to find η_0. You will need to solve eq. (4), which you can do any way you like, either by plotting dr/dt, or by using or writing software that can solve the equation, or as a last resort by trial an error trying different values of η by hand.

   Verify that your result indeed satisfies $\pi < \eta_0 < 2\pi$. *Hint:* you life will probably be easier of you rewrite eq. (4) in terms of the dimensionless quantity $\zeta = t_0(dr/dt)_0/r_0$.

(g) [5 points] Given η_0, r_0, and t_0, find the values of a and P. What was our maximum separation from M31? How long from now will we collide with M31? How does this compare to the remaining lifespan of the Sun (about 5 Gyr)?

(h) [5 points] Given a and P, find the total mass M of our two galaxies in solar masses.

  i. How does this compare to the Milky Way mass out to the LMC that you found in Problem Set 5? If the Milky Way luminosity is $L_B \simeq 2\times10^{10} L_\odot$, find M/L_B, compare with the solar neighborhood value, and comment.

  ii. How does M compare to M31's mass out to 50 kpc, if the rotation curve is flat with $v_c = 300$ km/s? If the M31 luminosity is $L_B \simeq 2.7 \times 10^{10} L_\odot$, find M/L_B, compare with the solar neighborhood value, and comment.

(i) [5 points] This analysis is known as Local Group timing. Why is *time* important here? In other words, why does the age of the universe (which we use for the age of the MW-M31 system) so important in this problem?

3. [2 bonus points] *Identify yourself!* To help your instructor connect names and faces, go to the course Compass site and find yourself in the class portrait.