Astronomy 501: Radiative Processes

Lecture 10

Sept 19, 2018

Announcements:

- **Problem Set 3** due Friday at start of class

Last time:

isotropic coherent scattering *Q: what's that? transfer eq?*

random walk *Q: what's that? rms progress after* N *steps?*

Combined Scattering and Absorption

generally, matter can both scatter and absorb photons
transfer equation must include both
for *coherent isotropic scattering* of *thermal radiation*

$$\frac{dI_\nu}{ds} = -\alpha_\nu(I_\nu - B_\nu) - \varsigma_\nu(I_\nu - J_\nu) \quad (1)$$

giving a source function

$$S_\nu = \frac{\alpha_\nu B_\nu + \varsigma_\nu J_\nu}{\alpha_\nu + \varsigma_\nu} \quad (2)$$

a *weighted average* of the two source functions

thus we can write

$$\frac{dI_\nu}{ds} = -(\alpha_\nu + \varsigma_\nu)(I_\nu - S_\nu) \quad (3)$$

with **extinction coefficient** $\alpha_\nu + \varsigma_\nu$

generalize mean free path:

$$\ell_{\mathrm{mfp},\nu} = \frac{1}{\alpha_\nu + \varsigma_\nu} \tag{4}$$

average distance between photon interactions

in random walk picture:
probability of step ending in *absorption*

$$\epsilon_\nu \equiv \alpha_\nu \ell_{\mathrm{mfp},\nu} = \frac{\alpha_\nu}{\alpha_\nu + \varsigma_\nu} \tag{5}$$

and thus step *scattering probability*

$$\varsigma_\nu \ell_{\mathrm{mfp},\nu} = \frac{\varsigma_\nu}{\alpha_\nu + \varsigma_\nu} = 1 - \epsilon_\nu \tag{6}$$

also known as **single scattering albedo**

source function:

$$S_\nu = \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu \tag{7}$$

Random Walk with Scattering and Absorption

in *infinite medium*: every photon created is eventually absorbed
typical absorption path $\ell_{\text{abs},\nu} = 1/\alpha_\nu$
typical number of scattering events until absorption is

$$N_{\text{scat}} = \frac{\ell_{\text{abs},\nu}}{\ell_{\text{mfp},\nu}} = \frac{\varsigma_\nu + \alpha_\nu}{\alpha_\nu} = \frac{1}{\epsilon_\nu} \tag{8}$$

so typical distance traveled between creation and absorption

$$\ell_* = \sqrt{N_{\text{scat}}}\ell_{\text{mfp},\nu} = \sqrt{\ell_{\text{abs},\nu}\ell_{\text{mfp},\nu}} = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \varsigma_\nu)}} \tag{9}$$

diffusion/thermalization length or *effective mean free path*

What about a *finite medium* of size s?
define optical thicknesses $\tau_{\text{scat}} = \varsigma_\nu s$, $\tau_{\text{abs}} = \alpha_\nu \, s$
and $\tau_* = s/\ell_* = \tau_{\text{scat}}^{1/2}(\tau_{\text{scat}} + \tau_{\text{abs}})^{1/2}$

Q: expected behavior if $\tau_* \ll 1$*?* $\tau_* \gg 1$*?*

$\tau_* = s/\ell_*$: path in units of photon travel until absorption

$\tau_* \ll 1$: *effectively thin* or *translucent*
photons random walk by scattering, seen before absorption
luminosity of thermal source with volume V is

$$L_\nu \stackrel{\text{thin}}{=} 4\pi \alpha_\nu B_\nu V = 4\pi j_\nu(T) V \tag{10}$$

$\tau_* \gg 1$: *effectively tick*
thermally emitted photons scattered then absorbed before seen
expect $I_\nu \to S_\nu \to B_\nu$
rough estimate of luminosity of thermal source:
most emission from "last scattering" surface of area A
where photons travel $s = \ell_*$

$$L_\nu \stackrel{\text{thick}}{\approx} 4\pi \alpha_\nu B_\nu \ell_* A \approx 4\pi \sqrt{\epsilon_\nu}\, B_\nu A \tag{11}$$

Walking on the Sun?

the Sun in optical (peak emission) shows a sharp surface

`www: the optical Sun today`

yet the Sun is a gasball–no surface at all!
and gas density drops continuously – no edge

Q: so what's the deal?
Q: what determines apparent surface?
Q: order of magnitude estimate?
Q: how would you make the calculation rigorous?

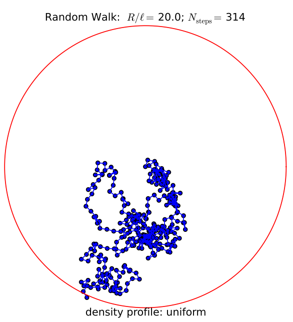
Stellar Photospheres

stellar photons are born in the deep interior
and scatter until they reach us

the photons that we see had *last scattering*
in the solar **photosphere**, roughly where

$$\ell_{\rm mfp|} \sim R_\odot$$

i.e., photosphere is where $\tau_{\rm sc}(R_\odot) \sim 1$



a more rigorous calculation studies the scattering in detail
e.g., Eddington approximation gives $\tau = 2/3$

Q: how are things different in solar interior?

Life Inside a Star

In stars:

- nuclear reactions create energy and γ rays deep in the interior (core)
- the energy and radiation escape to the surface after many interactions

How does this occur?

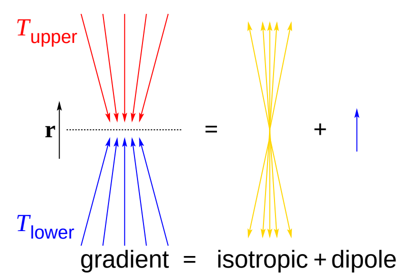
Consider a point at stellar radius r with temperature $T(r)$ having blackbody radiation at T, and matter

Q: what is intensity pattern (i.e., over solid angle) if T is uniform?
Q: what is the pattern more realistically?
Q: what drives the outward energy flow? what impedes it?
Q: relevant length scale(s) for radiation flow?

if $T(r)$ uniform and has *no gradient*,
so are blackbody intensity B and flux T
$\longrightarrow$ no net flow of radiation!

but in real stars: *T decreases with r*

so at r:



- intensity from below greater than from above
- drive net flux outwards
- impeded by scattering and absorption on scale $\ell_{\text{mfp},\nu} = (\alpha_\nu + \varsigma_\nu)^{-1}$
- generally $\ell_{\text{mfp},\nu} \ll r$: over this scale, see radiation as *mostly isotropic* with *small dipole*

Radiative Diffusion: Sketch Rosseland Approximation

given a small temperature dipole, expect *net radiation flux*

$$F_\nu^{\rm net} \sim -\pi \Delta B_\nu \sim -\pi \left[B_\nu(T_{r+\delta r}) - B_\nu(T_r) \right] \tag{12}$$
$$= -\pi \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial r} \delta r \tag{13}$$
$$\sim -\pi \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial r} \ell_{\rm mfp,\nu} \tag{14}$$

So the total flux $F = \int F_\nu^{\rm net} d\nu$ has

$$F = -\frac{4}{3}\pi \frac{\partial_T B}{\alpha_{\rm R}} \partial_r T \tag{15}$$

- $\vec{F} \propto -\nabla T$: diffusion flux! requires gradient!
- average over ν gives **Rosseland mean absorption coefficient**

$$\frac{1}{\alpha_{\rm R}} = \frac{\int (\alpha_\nu + \varsigma_\nu)^{-1} \partial_T B_\nu \, d\nu}{\int \partial_T B_\nu \, d\nu} \tag{16}$$

effective mean free path, weighted by Planck derivative

Radiative Flux in the Rosseland Approximation

using Rosseland mean, the (total) photon energy flux is

$$F(z) = -\frac{16\sigma T^3}{3\alpha_{\mathrm{R}}}\frac{\partial T}{\partial z} \tag{17}$$

Rosseland approximation to radiative flux

Q: what if T uniform? decreasing upwards? implications for stars?

Note:

- whenever energy (heat) flux $\vec{F} = -\chi \nabla T$
  coefficient χ is the *heat conductivity*
- in the presence of a heat flux, thermal energy density changes:

$$\partial_t u = -\nabla \cdot \vec{F} \tag{18}$$

a continuity equation, i.e., local statement of energy conservation
for radiation, $u = u(T)$, so $\partial_t T \sim D\nabla^2 T$: a diffusion equation!

in stars, energy must be transported from interior
where it is created by thermonuclear reactions
upwards until it is radiated to space

in regions when temperature gradient $\partial_z T$ not too large
radiative diffusion is the mechanism for energy transport
i.e., photons random walk their way out of the star

- typical solar photon is millions of years old
- unlike neutrinos which are minutes old

photon *luminosity* in interior radius r is

$$L(r) = 4\pi r^2 \; F(r) = -4\pi r^2 \frac{16\sigma T^3}{3\alpha_{\rm R}} \frac{\partial T}{\partial r} \tag{19}$$

solar temperature drops with radius, $\partial_z T < 0$,
so $L > 0$: energy flows outwards!

Classical Electromagnetic Radiation

Electromagnetic Forces on Particles

Consider *non-relativistic classical particle*
with mass m, charge q and velocity $\vec{v}$

under an electric field $\vec{E}$ and magnetic field $\vec{B}$
the particle feels a **force**

$$\vec{F} = q\ \vec{E} + q\ \frac{\vec{v}}{c} \times \vec{B} \tag{20}$$

sums Coulomb and Lorentz forces
units: cgs throughout; has nice property that $[E] = [B]$

power supplied by EM fields to charge

$$\frac{dU_{\text{mech}}}{dt} = \vec{v} \cdot \vec{F} = q\ \vec{v} \cdot \vec{E} = \frac{d}{dt}\frac{mv^2}{2} \tag{21}$$

no contribution from $\vec{B}$: "magnetic fields do no work"

Q: what if smoothly distributed charge density and velocity field?

Electromagnetic Forces on Continuous Media

consider a medium with charge density ρ_q
and current density $\vec{j} = \rho_q \vec{v}$

by considering an "element" of charge $dq = \rho_q \; dV$
we find **force density**, defined via $d\vec{F} = \vec{f} \; dV$:

$$\vec{f} = \rho_q \; \vec{E} + \frac{\vec{j}}{c} \times \vec{B} \tag{22}$$

and a **power density** supplied by the fields

$$\frac{\partial u_{\text{mech}}}{\partial t} = \vec{j} \cdot \vec{E} \tag{23}$$

note: if medium is a collection of point sources $q_i, \vec{r}_i, \vec{v}_i$

$$\rho_q(\vec{r}) = \sum_i q_i \; \delta(\vec{r} - \vec{r}_i) \tag{24}$$

and current density is

$$\vec{j}(\vec{r}) = \sum_i q_i \; \vec{v}_i \; \delta(\vec{r} - \vec{r}_i) \tag{25}$$

Maxwell's Equations

Maxwell relates fields to charge and current distributions

in the absence of dielectric media ($\epsilon = 1$)
or permeable media ($\mu = 1$):

$$\begin{aligned} \nabla \cdot \vec{E} &= 4\pi\rho_q && \text{Coulomb's law} \\ \nabla \cdot \vec{B} &= 0 && \text{no magnetic monopoles} \\ \nabla \times \vec{E} &= -\tfrac{1}{c}\partial_t \vec{B} && \text{Faraday's law} \\ \nabla \times \vec{B} &= \tfrac{4\pi}{c}\vec{j} + \tfrac{1}{c}\partial_t \vec{E} && \text{Ampère's law} \end{aligned} \tag{26}$$

take divergence of Ampère

$$\partial_t \rho_q + \nabla \cdot \vec{j} = 0 \tag{27}$$

conservation of charge!

now can rewrite power exerted by fields on charges
in terms of fields only *Q: how?*

Field Energy

Power density exerted by fields on charges

$$\frac{\partial u_{\text{mech}}}{\partial t} = \vec{j} \cdot \vec{E} = \frac{1}{4\pi}\left(c\nabla \times \vec{B} - \partial_t \vec{E}\right) \cdot \vec{E} \tag{28}$$

with clever repeated use of Maxwell,
can recast in this form:

$$\frac{\partial u_{\text{fields}}}{\partial t} + \nabla \cdot \vec{S} = -\frac{\partial u_{\text{mech}}}{\partial t} \tag{29}$$

Q: physical significance of eq. (29)?

energy change per unit time

$$\frac{\partial u_{\text{fields}}}{\partial t} + \nabla \cdot \vec{S} = -\frac{\partial u_{\text{mech}}}{\partial t} \tag{30}$$

reminiscent of $\partial_t \rho_q + \nabla \cdot \vec{j} = 0$

$\rightarrow$ an expression of local conservation of energy
where the mechanical energy acts as source/sink

identify **electromagnetic field energy density**

$$u_{\text{fields}} = \frac{E^2 + B^2}{8\pi} \tag{31}$$

i.e., $u_E = E^2/8\pi$, and $u_B = B^2/8\pi$

and **Poynting vector** is *flux of EM energy*

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \tag{32}$$

this is huge for us ASTR 501 folk! EM flux!
Q: when zero? nonzero? direction?

Electromagnetic Waves

in vacuum ($\rho_q = 0 = \vec{j}$), and in Cartesian coordinates Maxwell's equations imply (PS3):

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = 0 \tag{33}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} = 0 \tag{34}$$

both fields satisfy a **wave equation**

wave equation invites **Fourier transform** of fields:

$$\vec{E}(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int d^3\vec{r}\ dt\ \ \vec{E}(\vec{x},t)\ e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{35}$$

inverse transformation:

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2}\int d^3\vec{k}\ d\omega\ \ \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{36}$$

note symmetry between transformation (but sign flip in phase!)

original real-space field can be expressed as

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2} \int d^3\vec{k}\ d\omega\ \ \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{37}$$

expansion in *sum of Fourier modes* with

- **wavevector** $\vec{k}$
  magnitude $k = 2\pi/\lambda$, direction $\hat{n} = \vec{k}/k$
- **angular frequency** $\omega = 2\pi\ \nu$

apply wave equation to Fourier expansion:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = -\frac{1}{(2\pi)^2 c^2} \int d^3\vec{k}\ d\omega\ (c^2k^2 - \omega^2)\ \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{38}$$

$$= 0 \tag{39}$$

for notrivial solutions with $\vec{E} \neq 0$,
this requires $\omega^2 = c^2k^2$, or **vacuum dispersion relation**

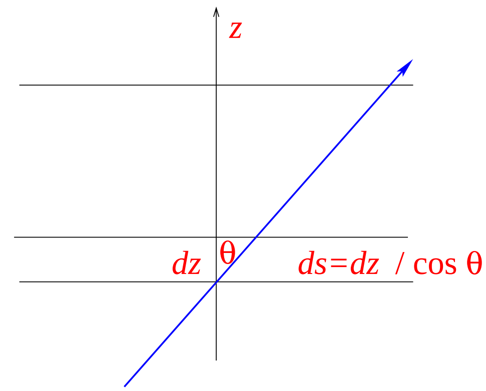
$$\omega = ck \tag{40}$$

i.e., wave solutions require constant phase velocity $v_\phi = \omega/k = c$

Director's Cut Extras

Rosseland Approximation in Detail

Imagine a **plane-parallel medium**:
n, ρ, T only depend on z
Think: interior of a star



photon propagation depends only on angle θ
between path direction and $\hat{z}$ *Q: why? why not on* ϕ *too?*

change to variable $\mu = \cos\theta$, and note that
path element $ds = dz/\cos\theta = dz/\mu$, so

$$\mu \frac{\partial I_\nu(z,\mu)}{\partial z} = -(\alpha_\nu + \varsigma_\nu)(I_\nu - S_\nu) \tag{41}$$

note: deep inside a real star like the Sun, $\ell_* \sim 1 \text{ cm} \ll R_\star$
Q: implications?

$\ell_* \sim 1$ cm $\ll R_\star$: rapid thermalization, damping of anisotropy

expect stellar interior to have intensity field that

- changes slowly compared to mean free path
- is nearly isotropic

so to *zeroth order* in ℓ_*, transfer equation

$$I_\nu = S_\nu - \mu \ell_* \frac{\partial I_\nu(z,\mu)}{\partial z} \tag{42}$$

gives

$$I_\nu^{(0)} \approx S_\nu^{(0)}(T) \tag{43}$$

this is angle-independent, so: $J_\nu^{(0)} = S_\nu^{(0)}$ and $I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$

Iterate to get *first-order approximation*

$$I_\nu^{(1)} \approx S_\nu^{(0)} - \mu \ell_* \partial_z I_\nu^{(0)} = B_\nu - \frac{\mu}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \tag{44}$$

what angular pattern does this intensity field have? why?

to first order, intensity pattern

$$I_\nu^{(1)} \approx S_\nu^{(0)} - \mu \ell_* \partial_z I_\nu^{(0)} = B_\nu - \frac{\mu}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \tag{45}$$

i.e., a dominant isotropic component plus
small correction $\propto \mu = \cos\theta$: a *dipole!*
if T decreases with z, then $\partial_z B_\nu < 0$, and so
intensity brighter downwards (looking into hotter region)

use this find **net specific flux along z**

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu) \ \cos\theta \ d\Omega = 2\pi \int_{-1}^{+1} I_\nu^{(1)}(z,\mu) \ \mu \ d\mu \tag{46}$$

only the *anisotropic* piece of $I_\nu^{(0)}$ of survives *Q: why?*

$$F_\nu(z) = -\frac{2\pi}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \int_{-1}^{+1} \mu^2 \ d\mu \tag{47}$$

$$= -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)} \partial_z B_\nu \tag{48}$$

net specific flux along z

$$F_\nu(z) = -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)}\partial_z B_\nu = -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)}\partial_T B_\nu \; \partial_z T \qquad (49)$$

since $B_\nu = B_\nu(T)$

total integrated flux

$$F(z) = \int F_\nu(z)\; d\nu = -\frac{4\pi}{3}\partial_z T \int (\alpha_\nu + \varsigma_\nu)^{-1} \frac{\partial B_\nu}{\partial T}\; d\nu \qquad (50)$$

to make pretty, note that

$$\int \partial_T B_\nu \; d\nu = \partial_T \int B_\nu \; d\nu = \partial_T B(T) = \frac{4\pi\sigma T^3}{\pi} \qquad (51)$$

and define **Rosseland mean absorption coefficient**

$$\frac{1}{\alpha_{\mathrm{R}}} = \frac{\int (\alpha_\nu + \varsigma_\nu)^{-1} \partial_T B_\nu \; d\nu}{\int \partial_T B_\nu \; d\nu} \qquad (52)$$

average effective mean free path, weighted by Planck derivative