# **Astronomy 501: Radiative Processes**

Lecture 11

Sept 21, 2018

Announcements:

- **Problem Set 3** due at end of the day. Ish.
- **Problem Set 4** due next Friday

Last time:

combining scattering and absorption

Today: changing gears

- heretofore, photon=quantum picture of EM radiation
- now, (re)visit classical picture of EM fields and waves

  *Q: regime of applicability?*

  *Q: classical force on charge $q$ with velocity $\vec{v}$?*

  *Q: power supplied by EM fields to the charge?*

# Classical Electromagnetic Radiation

# Electromagnetic Forces on Particles

Consider *non-relativistic classical particle*
with mass $m$, charge $q$ and velocity $\vec{v}$

in an electric field $\vec{E}$ and magnetic field $\vec{B}$
the particle feels Coulomb and Lorentz **forces**

$$\vec{F} = q \ \vec{E} + q \ \frac{\vec{v}}{c} \times \vec{B} \tag{1}$$

**units: cgs throughout**; has nice property that $[E] = [B]$
ugly SI equations in Extras below

**power supplied** by EM fields to charge

$$\frac{dU_{\text{mech}}}{dt} = \vec{v} \cdot \vec{F} = q \ \vec{v} \cdot \vec{E} = \frac{d}{dt}\frac{mv^2}{2} \tag{2}$$


no contribution from $\vec{B}$: "magnetic fields do no work"

*Q: what if smoothly distributed charge density and velocity field?*

# Electromagnetic Forces on Continuous Media

consider a medium with charge density $\rho_q$
and current density $\vec{j} = \rho_q \vec{v}$

by considering an "element" of charge $dq = \rho_q \; dV$
we find **force density**, defined via $d\vec{F} = \vec{f} \; dV$:

$$\vec{f} = \rho_q \; \vec{E} + \frac{\vec{j}}{c} \times \vec{B} \tag{3}$$

and a **power density** supplied by the fields

$$\frac{\partial u_{\text{mech}}}{\partial t} = \vec{j} \cdot \vec{E} \tag{4}$$

note: if medium is a collection of point sources $q_i, \vec{r}_i, \vec{v}_i$

$$\rho_q(\vec{r}) = \sum_i q_i \; \delta(\vec{r} - \vec{r}_i) \tag{5}$$

and current density is

$$\vec{j}(\vec{r}) = \sum_i q_i \; \vec{v}_i \; \delta(\vec{r} - \vec{r}_i) \tag{6}$$

forces control particle responses to fields
now need equations for fields themselves!

*Q: sources of electric fields? point source behavior?*
*Q: sources of magnetic fields? non-sources?*
*infinite wire behavior?*

# Maxwell's Equations

Maxwell relates fields to charge and current distributions

in the absence of dielectric media ($\epsilon = 1$)
or permeable media ($\mu = 1$):

$$
\begin{aligned}
\nabla \cdot \vec{E} &= 4\pi\rho_q & &\text{Coulomb's law} \\
\nabla \cdot \vec{B} &= 0 & &\text{no magnetic monopoles} \\
\nabla \times \vec{E} &= -\frac{1}{c}\partial_t \vec{B} & &\text{Faraday's law} \\
\nabla \times \vec{B} &= \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E} & &\text{Ampère's law}
\end{aligned}
\tag{7}
$$

imagine I know:
- fields $\vec{E}_1, \vec{B}_1$ arising from $\rho_1, \vec{j}_1$
- and fields $\vec{E}_2, \vec{B}_2$ arising from $\rho_2, \vec{j}_2$

now consider case of sources $\rho_1 + \rho_2$ and $\vec{j}_1 + \vec{j}_2$
*Q: what are the resulting fields? why?*

Maxwell's equations are linear in the fields and sources!
for example: if $\nabla \cdot \vec{E}_1 = \rho_1$ and $\nabla \cdot \vec{E}_2 = \rho_2$
then $\nabla \cdot (\vec{E}_1 + \vec{E}_2) = \rho_1 + \rho_2$

can show: same idea for currents

and thus: **superposition** holds!
*sum of sources* leads to fields that *sum solutions for each*

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*Q: divergence of Ampère?*

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E} \quad (8)$$

take divergence of Ampère

$$\partial_t \rho_q + \nabla \cdot \vec{j} = 0 \quad \text{continuity} \tag{9}$$

integrate over volume:

$$\frac{dQ}{dt} = \int \partial_t \rho_q \; dV = - \int \nabla \cdot \vec{j} \; dV \overset{\text{Gauss thm}}{=} - \int \vec{j} \cdot d\vec{A} = I_q \tag{10}$$

charge loss from volume is only due to current out

**conservation of charge!**

now can rewrite power exerted by fields on charges
in terms of fields only *Q: how?*

# Field Energy

Power density exerted by fields on charges

$$\frac{\partial u_{\mathrm{mech}}}{\partial t} = \vec{j} \cdot \vec{E} = \frac{1}{4\pi} \left( c\nabla \times \vec{B} - \partial_t \vec{E} \right) \cdot \vec{E} \tag{11}$$

with clever repeated use of Maxwell,
can recast in this form:

$$\frac{\partial u_{\mathrm{fields}}}{\partial t} + \nabla \cdot \vec{S} = -\frac{\partial u_{\mathrm{mech}}}{\partial t} \tag{12}$$

where $u_{\mathrm{fields}}$ and $\vec{S}$ depend only on the fields
and $u_{\mathrm{mech}}$ sums the particle (mechanical) energies

 *Q: physical significance of eq. (12)?*

energy change per unit time

$$\frac{\partial u_{\text{fields}}}{\partial t} + \nabla \cdot \vec{S} = -\frac{\partial u_{\text{mech}}}{\partial t} \tag{13}$$

reminiscent of $\partial_t \rho_q + \nabla \cdot \vec{j} = 0$
→ an expression of local conservation of energy
where the mechanical energy acts as source/sink

identify **electromagnetic field energy density**

$$u_{\text{fields}} = \frac{E^2 + B^2}{8\pi} \tag{14}$$

i.e., $u_E = E^2/8\pi$, and $u_B = B^2/8\pi$

and **Poynting vector** is *flux of EM energy*

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \tag{15}$$

this is huge for us ASTR 501 folk! EM flux!
*Q: when zero? nonzero? direction?*

# Maxwell in Vacuo

Now consider a **vacuum = no charges or currents**
Maxwell simplifies to

$$\nabla \cdot \vec{E} = 0 \tag{16}$$

$$\nabla \cdot \vec{B} = 0 \tag{17}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \tag{18}$$

$$\nabla \times \vec{B} = \frac{1}{c}\partial_t \vec{E} \tag{19}$$

*Q: are there trivial solutions?*

*Q: are there non-trivial solutions? why?*

*Q: what scales appear? what doesn't appear? implications?*

# Electromagnetic Waves

*in vacuum* ($\rho_q = 0 = \vec{j}$), and in Cartesian coordinates Maxwell's equations imply (PS3):

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = 0 \quad (20)$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} = 0 \quad (21)$$

*Q: why is this gorgeous and profound?*

*Q: natural description?*

vacuum Maxwell:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} \;=\; 0 \tag{22}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} \;=\; 0 \tag{23}$$

both fields satisfy a **wave equation**
i.e., both fields support (undamped) waves with *speed $c$*

simplest wave solutions: sinusoids
superposition: arbitrary wave is sum of sinusoids

wave equation invites **Fourier transform** of fields:

$$\vec{E}(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int d^3\vec{r}\; dt \;\; \vec{E}(\vec{x},t)\; e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{24}$$

inverse transformation:

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2}\int d^3\vec{k}\; d\omega \;\; \vec{E}(\vec{k},\omega)\; e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{25}$$

note symmetry between transformation (but sign flip in phase!)

# Director's Cut Extras

# Electromagentism in SI Units

Sadly, unit conversion in between SI and cgs is a stain on the otherwise beautiful subject of E&M

Here we summarize how the fundamental equations appear in SI units

Coulomb and Lorentz forces in SI

$$\vec{F} = q\vec{E} + q\ \vec{v} \times \vec{B} \qquad (26)$$

$$\vec{f} = \rho_q\ \vec{E} + \vec{j} \times \vec{B} \qquad (27)$$

**note this means that $E$ and $B$ have different units!**

Maxwell's equations in SI:

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{\rho_q}{\epsilon_0} && \text{Coulomb's law} \\
\nabla \cdot \vec{B} &- 0 && \text{no magnetic monopoles} \\
\nabla \times \vec{E} &= -\partial_t \vec{B} && \text{Faraday's law} \\
\nabla \times \vec{B} &= \mu_0 \vec{j} + \mu_0 \epsilon_0 \, \partial_t \vec{E} && \text{Ampère's law}
\end{aligned}
\tag{28}
$$

and we find that $\epsilon_0 \mu_0 = 1/c^2$

field energy density (note the ghastly lack of symmetry!)

$$u_{\text{fields}} = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \tag{29}$$

i.e., $u_E = \epsilon_0 E^2/2$, and $u_B = B^2/2\mu_0$

and **Poynting vector** is *flux of EM energy*

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \tag{30}$$