Astro 501: Radiative Processes
Lecture 12
Sept. 24, 2018

Announcements:

- **Problem Set 4** due Friday
- **Problem Set 3** extended to today

Last time: began classical EM radiation

Q: energy density?

Q: Poynting vector?

Electromagnetic Waves

in vacuum ($\rho_q = 0 = \vec{j}$), and in Cartesian coordinates Maxwell's equations imply (PS3):

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = 0 \quad (1)$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} = 0 \quad (2)$$

Q: why is this gorgeous and profound?

Q: natural description?

vacuum Maxwell:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} \;=\; 0 \tag{3}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} \;=\; 0 \tag{4}$$

both fields satisfy a **wave equation**
i.e., both fields support (undamped) waves with *speed c*

simplest wave solutions: sinusoids
superposition: arbitrary wave is sum of sinusoids

wave equation invites **Fourier transform** of fields:

$$\vec{E}(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int d^3\vec{r}\; dt \;\; \vec{E}(\vec{x},t)\; e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{5}$$

inverse transformation:

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2}\int d^3\vec{k}\; d\omega \;\; \vec{E}(\vec{k},\omega)\; e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{6}$$

note symmetry between transformation (but sign flip in phase!)

original real-space field can be expressed as

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2} \int d^3\vec{k}\ d\omega\ \ \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{7}$$

expansion in *sum of Fourier modes* with

- **wavevector** $\vec{k}$
  magnitude $k = 2\pi/\lambda$, propagation direction $\hat{n} = \vec{k}/k$
- **angular frequency** $\omega = 2\pi\ \nu$

apply wave equation to Fourier expansion:

$$\begin{aligned} \nabla^2\vec{E} - \frac{1}{c^2}\partial_t^2\vec{E} &= -\frac{1}{(2\pi)^2 c^2}\int d^3\vec{k}\ d\omega\ (c^2k^2-\omega^2)\ \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ &= 0 \end{aligned}$$

for non-trivial solutions with $\vec{E} \neq 0$,
this requires $\omega^2 = c^2k^2$, or **vacuum dispersion relation**

$$\omega = ck \tag{8}$$

i.e., wave solutions require constant phase velocity $v_\phi = \omega/k = c$

Maxwell and Fourier Modes

We have seen: wave equation demands $\omega = ck$
But Maxwell equations impose further constraints

Consider arbitrary Fourier modes
$\vec{E} = E_0 \; n e^{i(\vec{k}\cdot\vec{r}-\omega t)} \; \hat{a}_1$, and $\vec{B} = B_0 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)} \; \hat{a}_2$

Maxwell equations in vacuum impose conditions:
for example, Coulomb's law $\nabla \cdot \vec{E} = 0$ implies

$$\vec{k} \cdot \vec{E} = 0 \tag{9}$$

or equivalently $\hat{n} \cdot \hat{a}_1 = 0$

similarly, no monopoles requires

$$\vec{k} \cdot \vec{B} = 0 \qquad \hat{n} \cdot \hat{a}_2 = 0 \tag{10}$$

Q: what does this mean physically for the waves?

we found $\vec{k} \cdot \vec{E} = \vec{k} \cdot \vec{B} = 0$

$\longrightarrow$ propagation orthogonal to field vectors

$\Rightarrow$ *EM waves are transverse*

Faraday's law requires $\omega \vec{B} = c\vec{k} \times \vec{E}$, or

$$\vec{B} = \frac{c\vec{k}}{\omega} \times \vec{E} = \hat{n} \times \vec{E} \tag{11}$$

and Ampère's law gives $\vec{E} = -\hat{n} \times \vec{B}$

Q: what do these conditions imply for the waves?

Faraday's law gives $\vec{B} = \hat{n} \times \vec{E}$, so

$$\vec{E} \cdot \vec{B} = \vec{E} \cdot (\hat{n} \times \vec{E}) = 0 \tag{12}$$

$\Rightarrow$ $\vec{E}$ *and* $\vec{B}$ *are orthogonal to each other!*

Faraday also implies

$$|B|^2 = \hat{n}^2 |E|^2 - |\hat{n} \cdot \vec{E}|^2 = |E|^2 \tag{13}$$

using vector identity $(\hat{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \hat{a} \cdot \vec{c}\, \vec{b} \cdot \vec{d} - \hat{a} \cdot \vec{d}\, \vec{b} \cdot \vec{c}$

we have: $E_0 = B_0$: *field amplitudes are equal*

which in turn means: $\hat{a}_2 = \hat{n} \times \hat{a}_1$, and $\hat{a}_1 \cdot \hat{a}_2 = 0$
$\rightarrow$ $(\hat{n}, \hat{a}_1, \hat{a}_2)$ form an *orthogonal basis*

Monochromatic Plane Wave: Time Averaging

at a given point in space, field amplitudes vary
sinusoidally with time $\rightarrow$ energy density and flux also sinusoidal
but we are interested in timescales $\gg \omega^{-1}$:
$\rightarrow$ take *time averages*

Useful to use *complex* field amplitudes
then take *real part* to get physical component

handy theorem: for $A(t) = \mathcal{A}e^{i\omega t}$ and $B(t) = \mathcal{B}e^{i\omega t}$
i.e., same time dependence, then time-averaged products

$$\langle \mathrm{Re}A(t)\ \mathrm{Re}B(t) \rangle = \frac{1}{2}\mathrm{Re}(\mathcal{A}\mathcal{B}^*) = \frac{1}{2}\mathrm{Re}(\mathcal{A}^*\mathcal{B}) \tag{14}$$

Monochromatic Plane Wave: Energy, Flux

time-averaged Poynting flux amplitude

$$\langle S \rangle = \frac{c}{8\pi}\text{Re}(E_0 B_0^*) = \frac{c}{8\pi}|E_0|^2 = \frac{c}{8\pi}|B_0|^2 \tag{15}$$

relates intensity and field strength

time-averaged energy density

$$\langle u \rangle = \frac{|E_0|^2}{8\pi} = \frac{|B_0|^2}{8\pi} \tag{16}$$

and so $\left\langle \vec{S} \right\rangle = c \, \langle u \rangle \, \hat{n}$

Q: given wave direction $\vec{n}$, degrees of freedom in $\vec{E}, \vec{B}$?

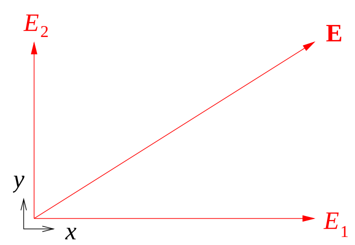
Polarization

EM waves propagating *in a particular direction* $\hat{n}$
must be transverse $\vec{k} \cdot \vec{E} = \hat{n} \cdot \vec{E} = 0$
$\rightarrow$ nonzero $\vec{E}$ *components lie in plane* $\perp$ to $\hat{n}$

two independent components

for convenience: choose coordinates where $\hat{n} = \hat{z}$
so fields are in transverse plane $x - y$
physical electric vector is *real part* of

$$\vec{E} = (E_1 \ \hat{x} + E_2 \ \hat{y}) \ e^{-i\omega t} \qquad (17)$$



complex amplitudes can be written

$$E_1 = \mathcal{E}_1 \ e^{i\phi_1} \quad E_2 = \mathcal{E}_2 \ e^{i\phi_2} \qquad (18)$$

Q: but wait–what about the magnetic field?

transverse electric field has two independent components
but once $\vec{E}$ determined, then $\vec{B} = \widehat{n} \times \vec{E}$
at every point along sightline $\widehat{n}$, magnetic $\perp$ electric
$\Rightarrow$ *no additional degrees of freedom for* $\vec{B}$

monochromatic plane wave *has two independent components*

consider plane at fixed $z = \widehat{n} \cdot \vec{r}$, say $z = 0$
imagine a detector in the plane, with $x - y$ axes

the two *physical* components of the $\vec{E}$ field
are *independent*, and evolve along the detector axes as

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2) \tag{19}$$

In our $x - y$ detector plane,
the independent $\vec{E}$ field component sare:

$$E_x = \mathcal{E}_1 \ \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \ \cos(\omega t - \phi_2) \tag{20}$$

with E_1, E_2 can take any values, and ϕ_1, ϕ_2 independent
but only difference $\phi_1 - \phi_2$ can be important
$\rightarrow$ a total of *3 independent parameters* describe the fields

Q: $\vec{E}$ time evolution if E_1 and E_2 can differ, but $\phi_1 - \phi_2 = 0$?
Q: same but $\phi_1 - \phi_2 = \pi$?

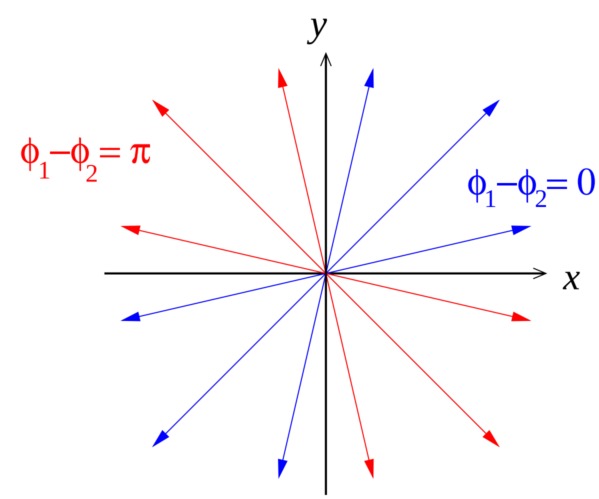
Linear Polarization

For $\phi_1 - \phi_2 = 0$, we have

$$E_x = \mathcal{E}_1 \ \cos(\omega t - \phi_1) \qquad E_y = \mathcal{E}_2 \ \cos(\omega t - \phi_1) = \frac{\mathcal{E}_2}{\mathcal{E}_1} E_x \tag{21}$$

fields share same sign and same sinusoidal time dependence
$\vec{E}$ sweeps out *line with positive slope* in $x - y$ plane
$\longrightarrow$ **linear polarization**

For $\phi_1 - \phi_2 = \pi$, fields share time dependence
but have opposite sign
$\longrightarrow$ linear polarization with negative slope

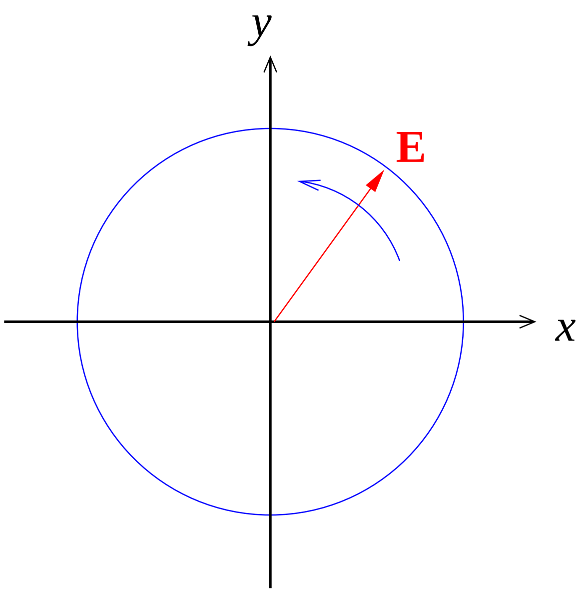


Q: what is $\vec{E}$ time dependence if
$\mathcal{E}_1 = \mathcal{E}_2$ *but* $\phi_1 - \phi_2 = \pi/2$*?* $-\pi/2$

Circular Polarization

if $\mathcal{E}_1 = \mathcal{E}_2$ but $\phi_1 - \phi_2 = \pi/2$

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_1 \sin(\omega t - \phi_1)$$

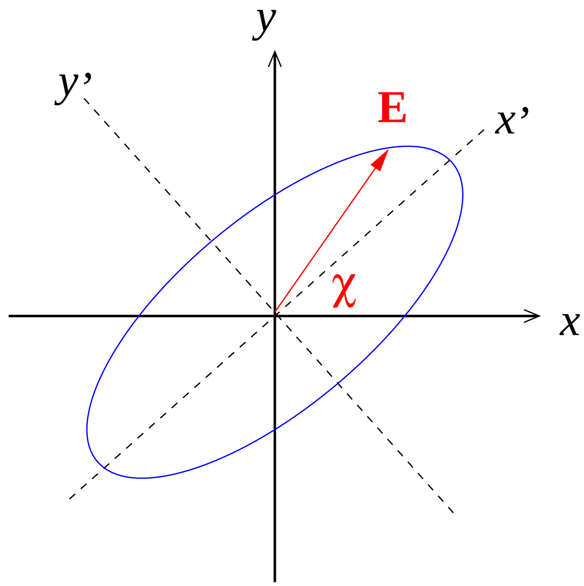


$\vec{E}$ sweeps counterclockwise circle
as seen approaching observer
$\Rightarrow$ **circular polarization**
Engineering: "*lefthanded*" circular polarization
$\longrightarrow$ but using righthand rule: *positive helicity*

if $\mathcal{E}_1 = \mathcal{E}_2$ but $\phi_1 - \phi_2 = -\pi/2$
$\longrightarrow$ "*righthand*" circular polarization, or *negative helicity*

in the most general case: $\mathcal{E}_1 \neq \mathcal{E}_2$ and $\phi_1 - \phi_2$ arbitrary
Q: what is $\vec{E}$ time dependence?

Elliptical Polarization



in the general case

$$E_x = \mathcal{E}_1 \ \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \ \cos(\omega t - \phi_2)$$

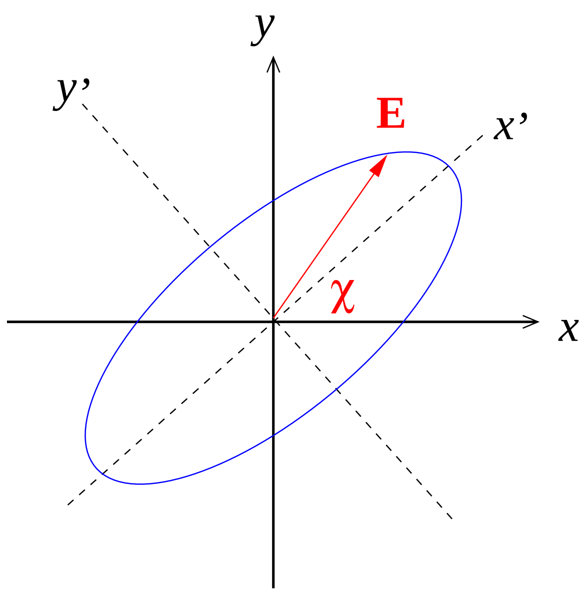
intuitively, blends linear and circular features:
→ **elliptical polarization**

ellipse *orientation* fixed by $\mathcal{E}_1 - \mathcal{E}_2$ difference
ellipse *eccentricity* and *helicity* fixed by $\phi_1 - \phi_2$ difference

in coordinates (x', y') rotated to align with *principal axes*

$$E'_x = \mathcal{E}_0 \ \cos\beta \ \cos(\omega t) \quad E'_y = -\mathcal{E}_0 \ \sin\beta \ \sin(\omega t)$$

for some $\beta \in [-\pi/2, +\pi/2]$
Q: evolution if $\beta > 0$*?*



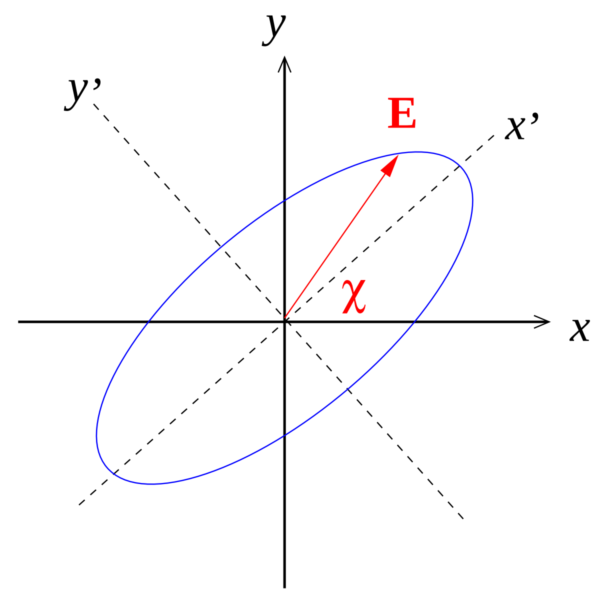
$$E'_x = \mathcal{E}_0 \;\cos\beta\; \cos(\omega t) \qquad E'_y = -\mathcal{E}_0 \;\sin\beta\; \sin(\omega t)$$

principle axes: $\mathcal{E}_0 \;\cos\beta$ and $\mathcal{E}_0 \;\sin\beta$

if $\beta \in [0, \pi/2]$: ellipse sweeps clockwise
$\longrightarrow$ "*righthanded*" elliptical polarization, *negative helicity*
if $\beta \in [-\pi/2, 0]$: *"lefthanded", positive helicity*

Q: what β(s) give complete linear polarization? circular?

we want to relate $x - y$ **field parameters** $\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$
to $x' - y'$ **principle axes parameters** $\mathcal{E}_0, \beta, \chi$



rotate $x - y$ components by angle χ

$$E_x = \mathcal{E}_0 (\cos\beta \cos\chi \cos\omega t + \sin\beta \sin\chi \sin\omega t)$$
$$E_y = \mathcal{E}_0 (\cos\beta \sin\chi \cos\omega t - \sin\beta \cos\chi \sin\omega t)$$

matching to, e.g., $E_x = \mathcal{E}_1 \cos(\omega t - \phi_1)$:

$$\mathcal{E}_1 \cos\phi_1 = \mathcal{E}_0 \cos\beta \cos\chi \quad (22)$$
$$\mathcal{E}_1 \sin\phi_1 = \mathcal{E}_0 \sin\beta \sin\chi \quad (23)$$
$$\mathcal{E}_2 \cos\phi_2 = \mathcal{E}_0 \cos\beta \sin\chi \quad (24)$$
$$\mathcal{E}_2 \sin\phi_2 = -\mathcal{E}_0 \sin\beta \cos\chi \quad (25)$$

Q: how can we determine polarization by intensity measurements with a polarimeters?

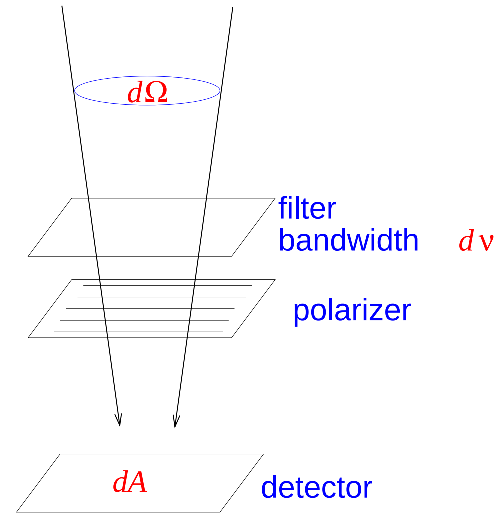
Introduce *polarizer*

can *rotate* polarizer:

$\longrightarrow$ measure I_x, I_y, and 45° rotated $I_{x'}, I_{y'}$

can use circular polarizers to measure

$\longrightarrow$ positive and negative circular polarization I_+, I_-



combine: **Stokes parameters**

$$I = I_x + I_y \quad (26)$$
$$Q = I_x - I_y \quad (27)$$
$$U = I_{x'} - I_{y'} \quad (28)$$
$$V = I_+ - I_- \quad (29)$$

Q: what physically is each? can more than one of Q, U, V be nonzero? what does that correspond to?

Q: range of values for Q? U? V? are they all independent?

Stokes Parameters

for *monochromatic waves*, Stokes parameters related to $\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$ and $\mathcal{E}_0, \beta, \chi$ bases:

$$I = \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 \tag{30}$$

$$Q = \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \tag{31}$$

$$U = 2\mathcal{E}_1\mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \tag{32}$$

$$V = 2\mathcal{E}_1\mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta \tag{33}$$

and thus

$$\mathcal{E}_0 = \sqrt{I} \tag{34}$$

$$\sin 2\beta = V/I \tag{35}$$

$$\tan 2\chi = U/Q \tag{36}$$

since wave has 3 independent parameters,
Stokes parameters must be *related*

$$I^2 = Q^2 + U^2 + V^2 \tag{37}$$

Quasi-Monochromatic Waves

natural light generally not a pure monochromatic wave
with a single, definite, complete state of polarization

rather: a *superposition* of components with many polarizations

consider wave with *slowly varying* amplitudes and phases

$$E_1(t) = \mathcal{E}_1(t)\ e^{i\phi_1(t)}\ ; \quad E_2(t) = \mathcal{E}_2(t)\ e^{i\phi_2(t)} \tag{38}$$

"slow": wave looks completely polarized on timescalse ω^{-1}
but amplitudes and phases drift over intervals $\Delta t \gg \omega^{-1}$
$\rightarrow$ polarization changes

but also wave is *no longer monochromatic*
frequency spread: "*bandwidth*" $\Delta\omega \sim 1/\Delta t \ll \omega$
$\rightarrow$ *quasi-monochromatic wave*

Q: effect on Stokes?

Stokes Parameters for Quasi-Monochromatic Light

real measurements represent averages over timescales during which polarization can change

Stokes parameters become averages

$$I = \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \right\rangle \quad (39)$$

$$Q = \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \right\rangle \quad (40)$$

$$U = \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = 2 \left\langle \mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \right\rangle \quad (41)$$

$$V = -i \left(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) = 2 \left\langle \mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \right\rangle \quad (42)$$

but for quasi-monochromatic waves

$$I^2 \geq Q^2 + U^2 + V^2 \quad (43)$$

- quasi-monochromatic polarization is still in general *elliptical*
- but drifts can reduce degree of polarization

$$I^2 \geq Q^2 + U^2 + V^2 \tag{44}$$

- maximum polarization when equality holds: *completely elliptically polarized*
- minimum when $Q = U = V = 0$: *unpolarized*
- arbitrary wave is *partially polarized*

useful to define *polarized* intensity

$$I_{\rm pol} = Q^2 + U^2 + V^2 \tag{45}$$

and since $I_{\rm pol} \leq I$, define fractional **degree of polarization**

$$\Pi \equiv \frac{I_{\rm pol}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} \tag{46}$$

note: can always decompose Stokes parameters

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - I_{\mathrm{pol}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} I_{\mathrm{pol}} \\ Q \\ U \\ V \end{pmatrix} \tag{47}$$

sum of unpolarized and polarized components

Superposition and Stokes

consider composite wave that is superposition of many independent waves

electric field components are given by **superposition**

$$E_1 = \sum_k E_1^{(k)} \quad ; \quad E_2 = \sum_k E_1^{(k)} \tag{48}$$

each term k of which has different phase

PS4: phases specified, can calculate sum explicitly

but generally, *phases are random*

so field products average out phases from different waves

$$\left\langle E_i E_j^* \right\rangle = \sum_k \sum_\ell \left\langle E_i^{(k)} E_j^{(\ell)*} \right\rangle = \sum_k \left\langle E_i^{(k)} E_i^{(k)*} \right\rangle \tag{49}$$

but due to this averaging, *Stokes parameters are additive*

$$I = \sum_k I^{(k)} \tag{50}$$

$$Q = \sum_k Q^{(k)} \tag{51}$$

$$U = \sum_k U^{(k)} \tag{52}$$

$$V = \sum_k V^{(k)} \tag{53}$$