Astro 501: Radiative Processes
Lecture 14
Sept. 28, 2018

Announcements:

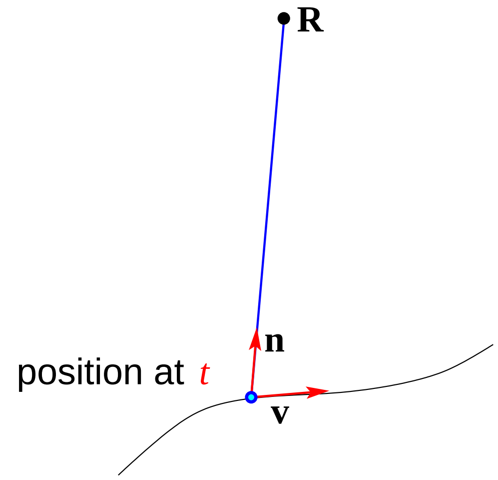
- **Problem Set 4** due now
- **Problem Set 5** due next Friday

Last time:

- polarization
- moving charges

Electrodynamics of Moving Charges: Strategy

point charge q:
position $\vec{R}(t)$
velocity $\vec{v} = \dot{\vec{R}} = \vec{\beta} c$
and acceleration $\vec{a} = \ddot{\vec{R}} = c \; d\vec{\beta}/dt = c\dot{\vec{\beta}}$



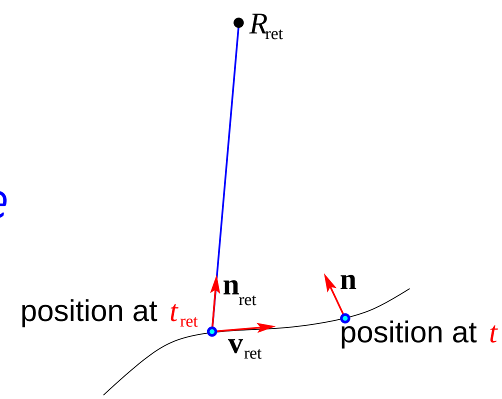
Maxwell sources:
charge density $\rho(\vec{x}) = q \; \delta(\vec{x} - \vec{R})$, current density $\vec{j} = \rho \vec{v}$

Procedure (see R&L and Extras for more):
0. Use full Special Relativity
1. write EM fields as derivatives of *4-potential* $(\phi, \vec{A})$
2. *Maxwell* $\rightarrow$ 2nd-order equations ∂^2potential $=$ source
3. solve for fields given above source terms

Electrodynamics of Moving Charges: Results

A careful calculation, and a lot of algebra, gives
an exact formula for the field of a moving point charge

$$\vec{E}(\vec{R},t) = q \left[\frac{(\hat{n} - \vec{\beta})(1-\beta^2)}{\kappa^3 R^2} \right]_{\rm ret} + \frac{q}{c} \left[\frac{\hat{n}}{\kappa^3 R} \times \left\{ (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}} \right\} \right]_{\rm ret}$$



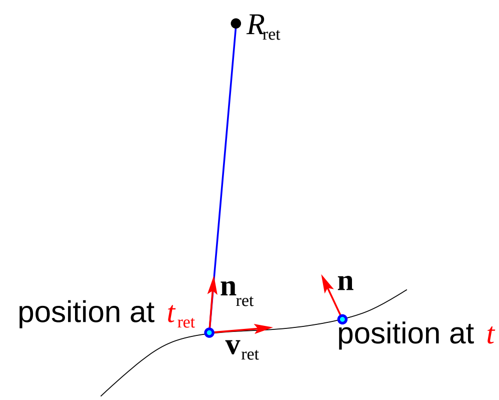
where $\kappa = 1 - \hat{n} \cdot \hat{\beta}$
and "ret" = particle position at *retarded time*
$t_{\rm ret} = t - R/c$

form is rich = complicated, but also complete and exact!
depends on charge position, velocity, and acceleration

for electric field

$$\vec{E}(\vec{R},t) = q\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{\rm ret} + \frac{q}{c}\left[\frac{\hat{n}}{\kappa^3 R}\times\left\{(\hat{n}-\vec{\beta}) \times \dot{\vec{\beta}}\right\}\right]_{\rm ret}$$

with $\kappa = 1 - \hat{n}\cdot\hat{\beta}$



magnetic field is $$\vec{B}(\vec{R},t) = \left[\hat{n}\times\vec{E}(\vec{R},t)\right]_{\rm ret}$$

Q: $\vec{E}$ result for charge at rest? $\vec{B}$?

Q: $\vec{E}$ for charge with with constant velocity?

Q: result at large R?

Electric "Velocity" Field

point source first term = "velocity field"

$$\vec{E}(\vec{R},t)_{\text{vel}} = q\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^2 R^2}\right]_{\text{ret}} \tag{1}$$

- depends only on position and velocity evaluated at a *past* location of the particle
- velocity field *not isotropic* if particle moving

displacement from retarded position $\vec{R}(t_{\text{ret}})$
to the field position $\vec{R}$ is $\hat{n}c(t-t_{\text{ret}})$
to the current particle position $\beta c(t-t_{\text{ret}})$
so $\vec{E}$ *points to current* position!

$\longrightarrow$ legal? yes! velocity constant, trajectory always "available"

Electric Acceleration Field

electric velocity field $\propto 1/R^2$
but other *acceleration* term $\propto \dot{v}_0$

$$\vec{E}(\vec{R},t)_{\text{accel}} = \frac{q}{c}\left[\frac{\hat{n}}{\kappa^3 R} \times \left\{(\hat{n}-\hat{\beta}) \ \times \dot{\hat{\beta}}\right\}\right]_{\text{ret}} \tag{2}$$

drops with distance $\propto 1/R$: always larger at large R

for nonrelativistic motion, $\beta_0 = v_0/c \ll 1$,
and so to first order

$$\vec{E}(\vec{R},t)_{\text{accel}} \approx \left[\frac{q}{c^2 R}\hat{n} \times (\hat{n} \times \vec{a})\right]_{\text{ret}} \tag{3}$$

a huge result!

Q: if acceleration is linear, what is polarization?

at large distances

$$\vec{E}(\vec{R},t) \rightarrow \vec{E}(\vec{R},t)_{\mathrm{accel}} \approx \left[\frac{q}{c^2 R}\hat{n} \times (\hat{n} \times \vec{a})\right]_{\mathrm{ret}} \tag{4}$$

instantaneous $\vec{E}$ *direction* set by $\hat{a}$ and $\hat{n}$

if acceleration is linear $\rightarrow$ $\hat{a}$ fixed

then $\vec{E}$ lies within $(\hat{n}, \hat{a})$ plane $\rightarrow$ *100% linearly polarized*

using $\vec{B} \rightarrow \hat{n} \times \vec{E}_{\mathrm{accel}}$, the Poynting flux is

$$\vec{S} \approx \frac{c}{4\pi} E^2_{\mathrm{accel}}\ \hat{n} = \frac{q^2}{4\pi c^3 R^2} \left|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})\right|^2 \hat{n} \tag{5}$$

Q: noteworthy features?

the Poynting flux is

$$\vec{S} \approx \frac{q^2}{4\pi c^3 R^2} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2 \tag{6}$$

$S \propto R_{\rm ret}^{-2}$: *flux obeys inverse square law!*

Power per unit solid angle is

$$\frac{dP}{d\Omega} = R^2 \hat{n} \cdot \vec{S} \approx \frac{c}{4\pi} |R\vec{E}_{\rm accel}|^2 = \frac{q^2}{4\pi c^3} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2 \tag{7}$$

independent of distance! *Q: why did this have to be true?*

Q: in which directions is $dP/d\Omega$ largest? smallest?
Q: radiation pattern?

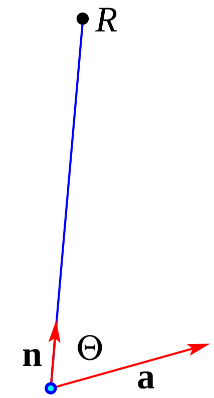
∞

Larmor Formula

Nonrelativistic charges radiate when accelerated!

Power per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2$$



define angle Θ *between* $\vec{a}$ *and* $\hat{n}$ via $\hat{n} \cdot \hat{\beta} = \cos\Theta$:

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2\Theta$$

a $\sin^2\Theta$ pattern!

$\rightarrow$ no radiation in direction of acceleration, maximum $\perp \vec{a}$
integrate over all solid angles: *total radiated power* is

$$P = \frac{q^2 a^2}{4\pi c^3} \int \sin^2\Theta d\Omega = \frac{2}{3} \frac{q^2}{c^3} a^2 \quad (8)$$

this will be our workhorse!

relates radiation to particle acceleration via $P \propto a^2$

An Accelerated Point Charge

consider a particle rapidly *deceleranted* from speed v to rest over time δt

consider a later time $t \gg \delta t$

Q: field configuration near *particle* ($r \ll ct$) *?*

Q: field configuration near *particle* ($r \gg ct$)*?*

Q: consequences?

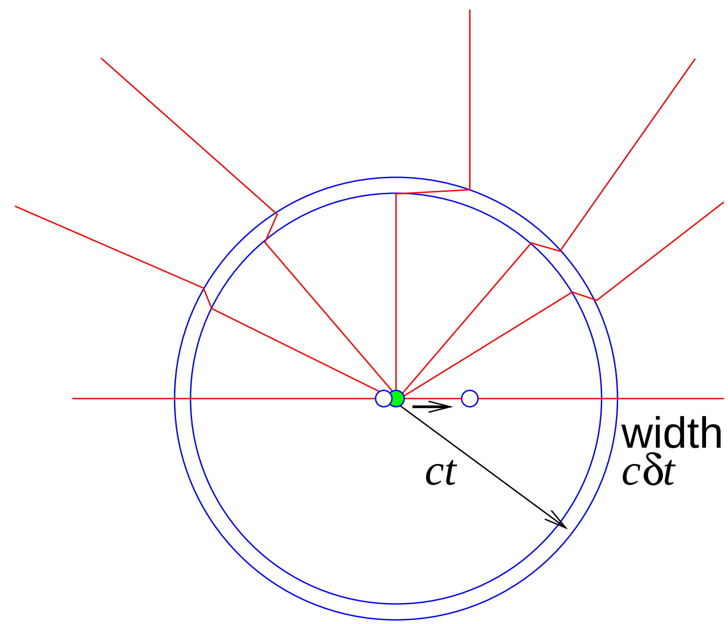
for fields track particle location expected for constant velocity

- nearby: $r \ll ct$, fields radial around particle at rest
- far away: $r \gg ct$: fields don't "know" particle has stopped
  $\rightarrow$ "anticipate" location displaced by ct from original particle
  radially oriented around this expected point

between the two regimes: $r = ct \pm c\delta t$

field lines must have "kinks" which

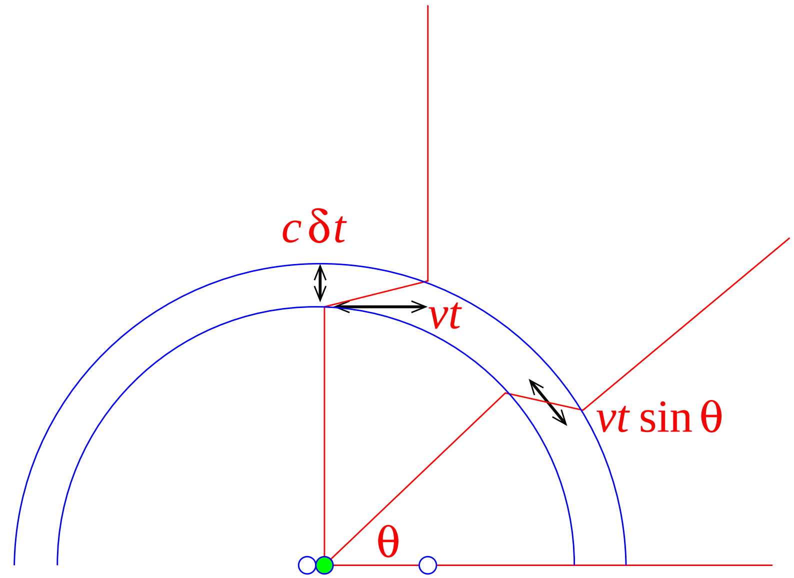
- have tangential field component
- tangential component is *anisotropic*
  and largest $\perp \vec{v}$



consider *vertical fieldline* $\perp \vec{v}$:

kink radial width $c\delta t$

kink tangential width $vt = (v/c)r$



tangential/radial ratio is $(v/\delta t)r/c^2$

but $v/\delta t = a$, average acceleration:

$\longrightarrow E_\perp/E_r = ar/c^2$

more generally, tangential width is

$vt \sin \Theta = (v/c)r \sin \Theta$

with angle Θ *between* $\vec{a}$ *and* $\hat{n}$

and so using Coulomb for E_r:

$$E_\perp = \frac{ar \sin \Theta}{c^2} E_r = \frac{qa}{c^2 r} \sin \Theta \tag{9}$$

this is huge! *Q: why?*

Q: relation to radiated flux?

We find acceleration leads to a propagating field perturbation that is **tangential = transverse!**
just what we expect for EM radiation

so we expect also a transverse $\vec{B}$ component, with

$$B_\perp = E_\perp = \frac{ar \sin \Theta}{c^2} E_r = \frac{qa}{c^2 r} \sin \Theta \tag{10}$$

and thus a radial Poynting vector with magnitude

$$S = \frac{c}{4\pi} E_\perp^2 = \frac{q^2 a^2}{4\pi c^3 r^2} \sin^2 \Theta \tag{11}$$

An Ensemble of Point Charges

Note: existence of kink and thus of radiation
demanded by combination of

- Gauss' law (field lines not created or destroyed in vacuum)
- finite propagation speed c

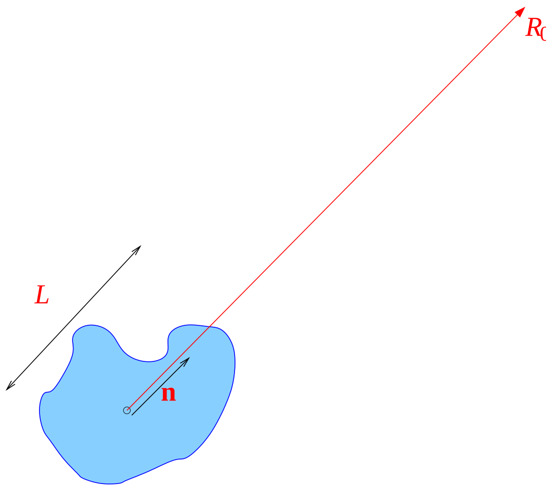
So far: field of a single point charge
Now: consider N particles, with q_i, $\vec{R}_i$, $\vec{v}_i = \dot{\vec{R}}_i$

Net $\vec{E}$ will be sum over all particles
Q: complications beyond "simple" bookkeeping?
Q: when will things simplify?

Approximate Phase Coherence

fields for each charge depend on it's *retarded time*
and these are different for each charge
$\longrightarrow$ leads to *phase differences* between particles
which we in general would have to track

When are phase differences not a problem?
When light-travel-time lags between particles
represent small phase differences



Let system size be L, and timescale for variations τ
if $\tau \gg L/c$, phase differences will be small

or: characteristic frequency is $\nu \sim 1/\tau$
so phase differences small if $c/\nu \gg L$, or $\lambda \gg L$
note that typical particle speeds $u \sim L/\tau$, so
phase coherence condition $\rightarrow u \ll c \rightarrow$ *nonrelativistic motion*

Dipole Approximation

so for **non-relativistic systems** we may ignore

- differences in time retardation, and
- the correction factor $\kappa = 1 - \hat{n} \cdot \vec{v}/c \to 1$

and thus we have

$$\vec{E}_{\mathrm{rad}} = \sum_i \frac{q_i}{c^2} \frac{\hat{n} \times (\hat{n} \times \vec{a}_i)}{R_i} \tag{12}$$

but the system has $R_i \approx R_0 \gg L$, and so

$$\vec{E}_{\mathrm{rad}} = \hat{n} \times \left(\frac{\hat{n}}{c^2 R_0} \times \sum_i q_i \vec{a}_i \right) = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \tag{13}$$

where the **dipole moment** is

$$\vec{d} = \sum_i q_i \vec{R}_i \tag{14}$$

for a non-relativistic dipole, we have

$$\vec{E}_{\rm rad} = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \quad (15)$$

this *dipole approximation* gives: power per unit solid angle

$$\frac{dP}{d\Omega} = \frac{\ddot{d}^{\,2}}{4\pi c^3} \sin^2 \Theta \quad (16)$$

and the total power radiated

$$\frac{dP}{d\Omega} = \frac{2}{3} \frac{\ddot{d}^{\,2}}{c^3} \quad (17)$$

consider a dipole that maintains the same orientation $\vec{d}$

$$E(t) = \ddot{d}(t)\frac{\sin\Theta}{c^2 R_0} \tag{18}$$

using Fourier transform of $d(t)$, we have

$$d(t) = \int e^{-i\omega t}\tilde{d}(\omega)\ d\omega \tag{19}$$

and so

$$\tilde{E}(\omega) = -\omega^2\tilde{d}(\omega)\frac{\sin\Theta}{c^2 R_0} \tag{20}$$

and thus the energy per solid angle and frequency is

$$\frac{dW}{d\Omega d\omega} = \frac{1}{c^3}\omega^4\ \left|\tilde{d}(\omega)\right|^2 \sin^2\Theta \tag{21}$$

and

$$\frac{dW}{d\omega} = \frac{8\pi}{3c^3}\omega^4\ \left|\tilde{d}(\omega)\right|^2 \tag{22}$$

- note the $\omega^4 \propto \lambda^{-4}$ dependence
- and $\tilde{d}(\omega)$: *dipole frequencies control radiation frequencies*

Director's Cut Extras

The Vector Potential

No-molopoles condition $\nabla \cdot \vec{B}$
strongly restricts $\vec{B}$ configurations

condition *automatically* satisfied if we write

$$\vec{B} = \nabla \times \vec{A} \tag{23}$$

guarantees zero divergence because, for *any* $\vec{A}$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \tag{24}$$

where $\vec{A}$ is the **vector potential**
Q: units of $\vec{A}$*?*

write Faraday's law in terms of $\vec{A}$:

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t(\nabla \times \vec{A}) \tag{25}$$

and so

$$\nabla \times \left(\vec{E} + \frac{1}{c}\partial_t \vec{A}\right) = 0 \tag{26}$$

strongly restricts $\vec{E}$ configurations

Q: how to automatically satisfy?

The Scalar Potential

Faraday with $\vec{A}$

$$\nabla \times \left(\vec{E} + \frac{1}{c} \partial_t \vec{A} \right) = 0 \quad (27)$$

vector field $\vec{E} + \frac{1}{c}\partial_t \vec{A}$ is curl-free

to automatically satisfy this, note that

$$\nabla \times (\nabla \phi) = 0 \quad (28)$$

curl of grad vanishes for any scalar field (=function) ϕ

define **scalar potential** via

$$\vec{E} = -\nabla \phi - \frac{1}{c} \partial_t \vec{A} \quad (29)$$

Q: units of ϕ?

Q: are $\vec{A}$ and ϕ unique? why?

Gauge Freedom

vector potential defined to give $\nabla \times \vec{A} = \vec{B}$
clearly if $\vec{A} \rightarrow \vec{A}' = \vec{A} +$ constant, $\vec{B} \rightarrow \vec{B}$
$\Rightarrow$ physical field unchanged

in fact: $\vec{B}$ unchanged for *any transformation*
$\vec{A} \rightarrow \vec{A}'$ which preserves $\nabla \times \vec{A}' = \vec{B}$:

$$\nabla \times (\vec{A}' - \vec{A}) = 0 \tag{30}$$

and thus there is no physical change if

$$\vec{A}' = \vec{A} + \nabla\psi \tag{31}$$

because $\nabla \times (\nabla\psi) = 0$ for any ψ
$\rightarrow$ *gauge invariance*

Q: what condition needed to keep $\vec{E}$ unchanged?

Gauge Invariance

the physical electric field has

$$\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t \vec{A} \tag{32}$$

and must remain the same when $\vec{A} \to \vec{A} + \nabla\psi$

but we have

$$\vec{E} \to \vec{E}' = -\nabla\phi - \frac{1}{c}\partial_t \vec{A}' \tag{33}$$

$$= -\nabla\left(\phi + \frac{1}{c}\partial_t \psi\right) - \frac{1}{c}\partial_t \vec{A} \tag{34}$$

Q: and so?

$$\vec{E} \to \vec{E}' \;=\; -\nabla\left(\phi + \frac{1}{c}\partial_t\psi\right) - \frac{1}{c}\partial_t\vec{A} \qquad (35)$$

and so to keep $\vec{E}' = \vec{E}$ requires

$$\phi \to \phi' = \phi - \frac{1}{c}\partial_t\psi \qquad (36)$$

the $\vec{E}, \vec{B}$ preserving mappings

$$(\phi, \vec{A}) \to (\phi, \vec{A}) + (\partial_t\psi/c, \nabla\psi) \qquad (37)$$

is a **gauge transformation**

a deep but also annoying property of electromagnetism
for our purposes, a useful but not unique choice

$$\nabla \cdot \vec{A} + \frac{1}{c}\partial_t\phi = 0 \qquad (38)$$

"Lorentz gauge"

Maxwell Revisited

express Maxwell in terms of potentials: Coulomb

$$-\nabla\cdot\left(\nabla\phi-\frac{1}{c}\partial_t\vec{A}\right) = -\nabla^2\phi-\frac{1}{c}\partial_t(\nabla\cdot\vec{A}) \tag{39}$$

$$= 4\pi\rho_q \tag{40}$$

and so in Lorentz gauge

$$\nabla^2\phi-\frac{1}{c^2}\partial_t^2\phi = -4\pi\rho_q \tag{41}$$

scalar potential satisfies a wave equation!
ϕ source is charge density ρ_q
changes in ϕ propagate at speed c

for *static* situation $\partial_t\phi = 0$, Poisson $\nabla^2\phi = -4\pi\rho_q$, and

$$\phi(\vec{r}) = \int d^3\vec{r}\,'\ \frac{\rho_q(\vec{r}\,')}{|\vec{r}\,'-\vec{r}|} \tag{42}$$

Q: solution for full wave equation?

Scalar Potential and Retarded Time

general solution to

$$\nabla^2 \phi - \frac{1}{c^2} \partial_t^2 \phi = -4\pi \rho_q \tag{43}$$

turns out to be

$$\phi(\vec{r}, t) = \int d^3\vec{r}\,' \; \frac{\rho_q(\vec{r}\,', t')}{|\vec{r}\,' - \vec{r}|} = \int d^3\vec{r}\,' \; \left[\frac{\rho_q}{|\vec{r}\,' - \vec{r}|} \right]_{\rm ret} \tag{44}$$

where source density $\rho_q(\vec{r}\,', t')$
is evaluated at **retarded time**

$$t' \equiv [t_{\rm ret}] = t - \frac{|\vec{r} - \vec{r}\,'|}{c} \tag{45}$$

$\rightarrow \phi$ "learns" about changes in charge density at $\vec{r}\,'$
only after signal propagation time $ct_{\rm prop} = |\vec{r}\,'|$

Maxwell and the Vector Potential

in terms of potentials, Ampère in Cartesian coords:

$$\nabla \times (\nabla \times \vec{A}) = \nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) \tag{46}$$

$$= \frac{4\pi}{c}\vec{j} + \frac{1}{c}\left(\nabla\phi + \partial_t \vec{A}\right) \tag{47}$$

so in Lorentz gauge

$$\nabla^2 \vec{A} - \frac{1}{c^2}\partial_t^2 \vec{A} = -\frac{4\pi}{c}\vec{j} \tag{48}$$

vector potential also satisfies a wave equation
source is current density $\vec{j}$

Q: solution?

each component A_i of vector potential satisfies

$$\nabla^2 A_i - \frac{1}{c^2}\partial_t^2 A_i = -\frac{4\pi}{c} j_i \tag{49}$$

formally identical to scalar potential equation
if we put $\phi \to A_i$ and $\rho_q \to j_i/c$

and thus we can import the solution:

$$A_i(\vec{r}, t) = \int d^3\vec{r'} \ \left[\frac{j_i}{|\vec{r'} - \vec{r}|}\right]_{\text{ret}} \tag{50}$$

$\to$ vector potential responds to current changes
after "retarded time" delay

Integral solutions for ϕ and $\vec{A}$ are huge!
 Q: why? what's the Big Deal?

Recipe for Electromagnetic Fields

our mission: find $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$
given charge $\rho_q(\vec{r}, t)$ and current $\vec{j}(\vec{r}, t)$ distributions

solution: first find potentials via

$$\phi(\vec{r}, t) = \int d^3\vec{r}' \left[\frac{\rho_q}{|\vec{r}' - \vec{r}|}\right]_{\rm ret} \quad (51)$$

$$\vec{A}(\vec{r}, t) = \int d^3\vec{r}' \left[\vec{j}|\vec{r}' - \vec{r}|\right]_{\rm ret} \quad (52)$$

from these, find fields via

$$\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t \vec{A} \quad (53)$$

$$\vec{B} = \nabla \times \vec{A} \quad (54)$$

ta da!

in the 3-D spatial integrals

$$\phi(\vec{r},t) = -\int d^3\vec{r}' \left[\frac{\rho_q}{|\vec{r}' - \vec{r}|}\right]_{\rm ret} \tag{55}$$

it is convenient (and pretty!) to recast as integrals over 4-D spacetime:

$$\phi(\vec{r},t) = -\int d^3\vec{r}'\ dt'\ \frac{\rho_q(\vec{r}',t')}{|\vec{r}' - \vec{r}|}\ \delta(t' - t + |\vec{r} - \vec{r}'|/c) \tag{56}$$

were the δ function enforces the retarded time condition

Q: What if charges are all pointlike?

Potentials from Point Charges

if N point charges, where ith charge q_i has trajectory with position $\vec{r}_i(t)$, and velocity $\vec{v}_i(t)$, then

$$\rho_q(\vec{r},t) = \sum_i q_i \ \delta^{(3)}(\vec{r}-\vec{r}_i) \tag{57}$$

$$\vec{j}(\vec{r},t) = \sum_i q_i \ v_i(t) \ \delta^{(3)}(\vec{r}-\vec{r}_i) \tag{58}$$

with Dirac δ-functions $\delta^{(3)}(\vec{r}-\vec{r}_i) = \delta(x-x_i)\ \delta(y-y_i)\ \delta(z-z_i)$

scalar potential due to *one charge* with $q_0, \vec{r}_0(t), \vec{v}_0(t)$ is

$$\phi(\vec{r},t) = q_0 \int d^3\vec{r}'\ dt'\ \frac{\delta^{(3)}(\vec{r}'-\vec{r}_0(t))}{|\vec{r}'-\vec{r}|}\ \delta(t'-t+|\vec{r}-\vec{r}'|/c) \tag{59}$$

space part of integral is easy

$$\phi(\vec{r},t) = q_0 \int dt'\ \frac{\delta\left(t'-t+|\vec{r}-\vec{r}_0(t')|/c\right)}{|\vec{r}-\vec{r}_0(t')|} \tag{60}$$

writing $\vec{R}(t') \equiv \vec{r} - \vec{r}_0(t')$
and $R(t') = |\vec{R}(t')|$, we have

$$\phi(\vec{r},t) = q_0 \int dt' \; \frac{\delta\left(t' - t + R(t')/c\right)}{R(t)} \tag{61}$$

and now the final δ function is nontrivial

math aside: fun properties of the δ function
$\delta(x)$ designed to give

$$\int f(y)\ \delta(y-x)\ dy = f(x) \tag{62}$$

but if δ argument is a function of the integration variable

$$\int f(y)\ \delta\left(g(x)\right)\ dy = \sum_{\text{roots}_j} \frac{f(g(x_j))}{|dg/dx|_{x_j}} \tag{63}$$

where root x_j is the jth solution to $y - g(x) = 0$

here: define $t'' = t' - t + R(t')/c$
then $dt'' = dt' + \dot{R}(t')/c\ dt'$

Liénard-Wiechert Potentials

for point source with arbitrary trajectory, we have

$$\phi(\vec{r},t) = \frac{1}{1 - \hat{n}\cdot\hat{\beta}_0(t_{\rm ret})}\frac{q_0}{R} \tag{64}$$

where $\hat{n} = \vec{r}/r$ and $\vec{\beta}_0(t) = \vec{v}_0(t)/c$

similarly, vector potential solution is

$$\vec{A}(\vec{r},t) = \frac{1}{1 - \hat{r}\cdot\hat{\beta}_0(t_{\rm ret})}\frac{q_0\vec{v}_0(\vec{r},t_{\rm ret})}{R(t_{\rm ret})} \tag{65}$$

these are the **Liénard-Wiechert potentials**

Q: equipotential surfaces $\phi = const$ *for stationary charge* $\vec{r}_0(t) = const$*?*
Q: for charge with $\vec{v}_0$ *large?*
Q: implications?

potential factor $\kappa \equiv [1 - \hat{n} \cdot \hat{\beta}]_{\rm ret}$ is

- directional,
- velocity dependent, such that
- *potential* $\propto 1/\kappa$ *enhanced along direction of charge motion* and *potential suppressed opposite direction of charge motion*

$\Rightarrow$ expect forward "beaming" effects!

But we want the EM fields, not just potentials,
so we need to evaluate

$$\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t \vec{A} \quad (66)$$

$$\vec{B} = \nabla \times \vec{A} \quad (67)$$

using the beautiful Liénard-Wiechert point-source potentials
where, $\phi = \phi[\vec{r}, t; \vec{r}_0(t), \vec{v}_0(t)]$ and $\vec{A} = \vec{A}[\vec{r}, t; \vec{r}_0(t), \vec{v}_0(t)]$

Q: what terms will appear in $\vec{E}$*?*