Astro 501: Radiative Processes
Lecture 15
October 1, 2018

Announcements:

- **Problem Set 5** due Friday

Last time:

- for charge with acceleration $\vec{a}$, viewed in direction $\hat{n}$
  Q: acceleration field $\vec{E}$ direction? $\vec{B}$?
- the glorious Larmor formula
  Q: expression for $dP/d\Omega$? angular pattern? P?
- dipole approximation

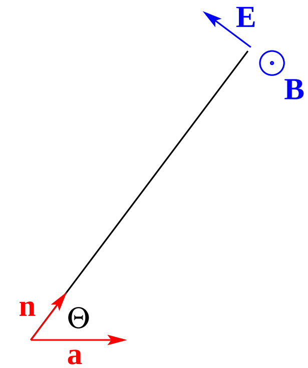
Q: when is it appropriate?

Q: what's the result?

Today: Thomson scattering

Larmor: acceleration $\vec{a}$, observer direction $\hat{n}$

power per unit solid angle is



$$\frac{dP}{d\Omega} \overset{\text{non-rel}}{=} \frac{q^2}{4\pi c^3} |\hat{n} \times (\hat{n} \times \vec{a})|^2 = \frac{q^2}{4\pi c^3} |-\vec{a} + (\hat{n} \cdot \vec{a})\hat{n}|^2$$

define angle Θ *between* $\vec{a}$ *and* $\hat{n}$ via $\hat{n} \cdot \hat{a} = \cos\Theta$:

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2\Theta \tag{1}$$

a $\sin^2\Theta$ pattern!

integrate over all solid angles: *total radiated power* is

$$P = \frac{2}{3}\,\frac{q^2}{c^3} a^2 \tag{2}$$

for a non-relativistic dipole, we have

$$\vec{E}_{\mathrm{rad}} = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \tag{3}$$

this *dipole approximation* gives: power per unit solid angle

$$\frac{dP}{d\Omega} = \frac{\ddot{d}^{\,2}}{4\pi c^3} \sin^2 \Theta \tag{4}$$

and the total power radiated

$$\frac{dP}{d\Omega} = \frac{2}{3}\frac{\ddot{d}^{\,2}}{c^3} \tag{5}$$

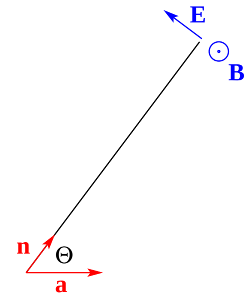
angular dependence is again $\sin^2 \Theta$

Q: what multipole is this?

Radiation from Accelerated Charges: Polarization

Polarization is electric field direction $\vec{E}$
where $\vec{E} \perp \vec{B} \perp \hat{n}$

Observationally: use polarizer which selects out
one of two polarization states $\hat{\epsilon}_1, \hat{\epsilon}_2$ in some (complex)
basis



e.g., if wave propagates in $\hat{n} = \hat{z}$ then
- xy polarization: $\epsilon_1 = \hat{x}$, $\epsilon_2 = \hat{y}$
- $x'y'$ polarizations: $\epsilon_1 = (\hat{x} + \hat{y})/\sqrt{2}$, $\epsilon_2 = (\hat{x} - \hat{y})/\sqrt{2}$
- circular polarization: $\epsilon_+ = (\hat{x} - i\hat{y})/\sqrt{2}$, $\epsilon_- = (\hat{x} + i\hat{y})/\sqrt{2}$

If (complex) electric vector is $\vec{E}$
Q: what passes through polarizer $\hat{\epsilon}_1$?

Q: how to find angular distribution $dP/d\Omega$ seen by polarizer $\hat{\epsilon}_1$?
Q: what about initially unpolarized radiation?

Complex electric vector is $\vec{E}$ can be written in some *polarization basis* $(\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{n} = \hat{k})$ as

$$\vec{E} = (\mathcal{E}_1 \hat{\epsilon}_1 + \mathcal{E}_2 \hat{\epsilon}_2)\, e^{i\vec{k}\cdot\vec{r} - i\omega t} \tag{6}$$

with complex amplitudes $\mathcal{E}_1$ and $\mathcal{E}_2$

the polarizer corresponding to $\hat{\epsilon}_1$ selects out this field component, i.e., the transmitted field amplitude is

$$E_1 = \hat{\epsilon}_1^* \cdot \vec{E} = \mathcal{E}_1 e^{i\vec{k}\cdot\vec{r} - i\omega t} \tag{7}$$

and so the angular distribution of power measured in *polarization state* $\hat{\epsilon}_1$ is

$$\left(\frac{dP}{d\Omega}\right)_{\rm pol,1} = \frac{c}{4\pi} |E_1|^2 = \frac{c}{4\pi} |\hat{\epsilon}_1^* \cdot \vec{E}|^2 \tag{8}$$

for *scattering of initially unpolarized* radiation: take average over possible initial polarizations

$$\left(\frac{dP}{d\Omega}\right)_{\rm unpol} = \frac{1}{2} \left[\left(\frac{dP}{d\Omega}\right)_{\rm pol,init1} \left(\frac{dP}{d\Omega}\right)_{\rm pol,init2} \right] \tag{9}$$

Thomson Scattering

Consider *monochromatic* radiation
linearly polarized in direction $\hat{\epsilon}_{\rm init}$
incident on a free, non-relativistic electron

because non-relativistic, we may ignore magnetic forces *Q: why?*

Q: equation of motion?

Q: and so?

Q: radiation pattern?

magnetic/electric force ratio $F_B/F_E \sim (v/c)B/E = v/c \ll 1$
and so we can ignore F_B

thus the force on the electron is

$$\vec{F} \approx -eE_0 \hat{\epsilon}_{\mathrm{init}} \cos \omega_0 t \tag{10}$$

and thus the electron has

$$\ddot{\vec{r}} = -\frac{e}{m_e} E_0 \hat{\epsilon}_{\mathrm{init}} \cos \omega_0 t \tag{11}$$

and so the dipole moment $\vec{d} = -e\vec{r}$ has

$$\ddot{\vec{d}} = \frac{e^2}{m_e} E_0 \hat{\epsilon}_{\mathrm{init}} \cos \omega_0 t \tag{12}$$

we can solve for the dipole moment

$$\vec{d} = -\frac{e^2 E_0}{m_e \omega_0^2} \hat{\epsilon}_{\mathrm{init}} \cos \omega_0 t \tag{13}$$

and thus the time-averaged power is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^4 E_0^2}{8\pi m_e^2 c^3} \sin^2 \Theta \tag{14}$$

$$\langle P \rangle = \frac{e^4 E_0^2}{3 m_e^2 c^3} \tag{15}$$

were Θ is angle between $\hat{n}$ and $\hat{a} = \hat{\epsilon}_{\mathrm{init}}$

Q: what's notable about these expressions?

Q: how could we disentangle intrinsic electron response?

Thomson Cross Section

time-averaged power

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^4 E_0^2}{8\pi m_e^2 c^3} \sin^2\Theta = \frac{e^4}{m_e^2 c^4} \sin^2\Theta \;\; \langle S \rangle \quad (16)$$

where time-averaged incident flux is $\langle S \rangle = cE_0^2/8\pi$

recall: **differential scattering cross section** can be defined as

$$\frac{d\sigma}{d\Omega} = \frac{\text{scattered power}}{\text{incident flux}} = \frac{dP/d\Omega}{\langle S \rangle} \quad (17)$$

$$= \frac{e^4}{m_e^2 c^4} \sin^2\Theta \quad (18)$$

integral **Thomson cross section** is

$$\sigma_\mathrm{T} \equiv \int \frac{d\sigma}{d\Omega} = \frac{8\pi}{3} \frac{e^4}{m_e^2 c^4} = \frac{8\pi}{3} r_0^2 = 0.665 \times 10^{-24} \text{ cm}^2 \quad (19)$$

with the *classical electron radius* $r_0 \equiv e^2/m_e c^2$

Thomson Appreciation

We have found the cross section for scattering of monochromatic, linearly polarized radiation on free electrons:

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{m_e^2 c^4} \sin^2 \Theta \qquad (20)$$

$$\sigma = \sigma_T = \frac{8\pi}{3} \frac{e^4}{m_e^2 c^4} \qquad (21)$$

Q: notable features?

Q: dependence (or lack thereof) on incident radiation?

plasmas will generally have ions as well as free electrons
Q: which is more important for Thomson scattering?

Q: under what conditions might our assumptions break down?

The Charms of Thomson

Thomson scattering is

- *independent of radiation frequency*
  implicitly assumes electron recoil negligible

$\rightarrow$ initial spectral *shape vs* ν is *unchanged*!

- example: Solar corona highly ionized, Thomson dominates
  Q: implications: spectrum/color? angular distribution?
  Q: how observe? `www: corona`

- $\sigma \propto 1/m^2$: *electron scattering larger than ions*
  by factor $(m_{\rm ion}/m_e)^2 \gg 10^6$!

- if electron recoil large, and/or electron relativistic
  assumptions break down, will have to revisit

if we measure polarization state $\hat{\epsilon}$,
Q: what is angular pattern of scattered radiation?

in measured = final polarization state $\hat{\epsilon}_{\mathrm{f}}$, find

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{m_e^2 c^4} \left| \hat{\epsilon}_{\mathrm{f}}^* \cdot \hat{\epsilon}_{\mathrm{init}} \right|^2 \tag{22}$$

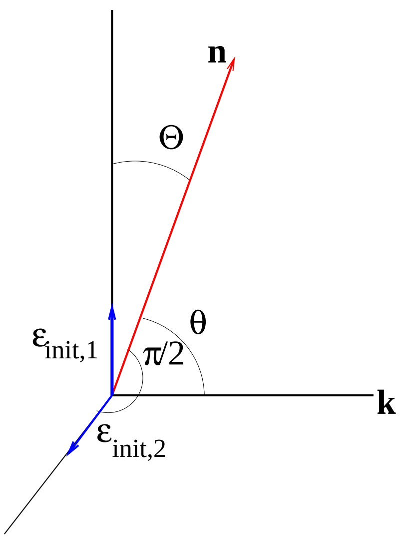
What if radiation is *unpolarized*?

Q: how can we use our result?

Thomson Scattering of Unpolarized Radiation

Using result for linear polarization
we can construct result for unpolarized radiation
by *averaging results for two orthogonal linear polarizations*

Geometry:
$\hat{n}$ is direction of scattered radiation
$\hat{\epsilon}_{\rm init} = \hat{k}$ direction of incident radiation
initial polarizations are both $\perp \hat{k}$
choose one polarization $\hat{\epsilon}_{\rm init,1}$ in $\hat{n} - \hat{k}$ plane
and the other $\hat{\epsilon}_{\rm init,2}$ orthogonal
to this plane and to $\hat{n}$



13 thus scatter initial polarization 1 by angle $\Theta = \pi/2 - \theta$
and an initial polarization 2 by angle $\pi/2$

thus scatter polarization 1 by angle $\Theta = \pi/2 - \theta$
and polarization 2 by angle $\pi/2$, and so

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = = \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_1 + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_2 \tag{23}$$

$$= \frac{r_0^2}{2}\left(1 + \sin^2\Theta\right) \tag{24}$$

$$= \frac{r_0^2}{2}\left(1 + \cos^2\theta\right) \tag{25}$$

which only depends on angle θ
between incident $\hat{k}$ and scattered $\hat{n}$ radiation direction

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{r_0^2}{2}\left(1 + \cos^2\theta\right) \tag{26}$$

- forward-backward asymmetry: $\theta \to -\theta$ invariance

- angular pattern: $\cos^2\theta \propto \cos 2\theta$ term

$\longrightarrow$ scattered radiation has has 180^0 periodicity

$\longrightarrow$ a "pole" every 90^0: **quadrupole**

- total cross section $\sigma_{\text{unpol}} = \sigma_{\text{pol}} = \sigma_{\text{T}}$

$\longrightarrow$ electron at rest has no preferred direction

- Polarization of scattered radiation

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta} \tag{27}$$

Q: what does this mean?

Thomson Scattering Creates Polarization

Thomson scattering of *initially unpolarized* radiation has

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta} \tag{28}$$

i.e., degree of polarization $P \neq 0$!

Thomson-scattered radiation is linearly polarized!

Quadrupole pattern in angle θ between $\hat{k}_{\text{init}}$ and $\hat{n}_{\text{scattered}}$

- 100% polarized at $\theta = \pi/2$
- 0% polarized at $\theta = 0, \pi$

classical picture: e^- as dipole antenna

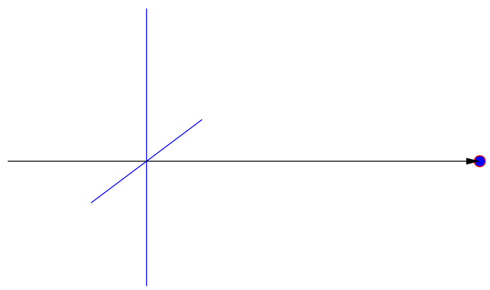
incident linearly polarized wave accelerates e^-

$\rightarrow \sin^2\Theta$ pattern, peaks at $\Theta = 0$, i.e., $\|\hat{\epsilon}_{\text{init}}$

Thompson Scattering: A Gut Feeling

Discussion swiped from Wayne Hu's website

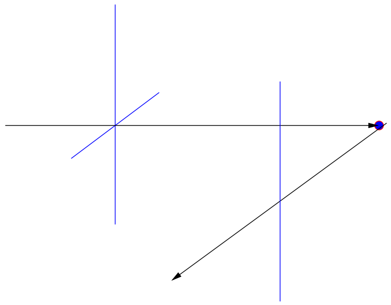
Consider a beam of unpolarized radiation
propagating in plane of sky, incident on an electron
think of as superposition of linear polarizations
  one along sightline, one in sky



Q: why is scattered radiation polarized?

Q: now what if unpolarized beams from opposite directions?

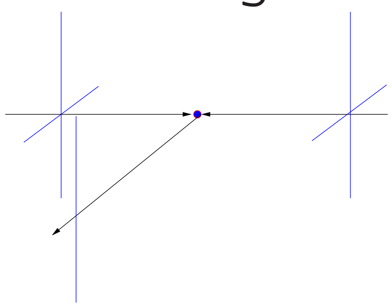
scattering of one unpolarized beam:



$\longrightarrow$ see radiation from e motion in sky plane

$\longrightarrow$ linear polarization!

scattering of two unpolarized beams in opposite directions:

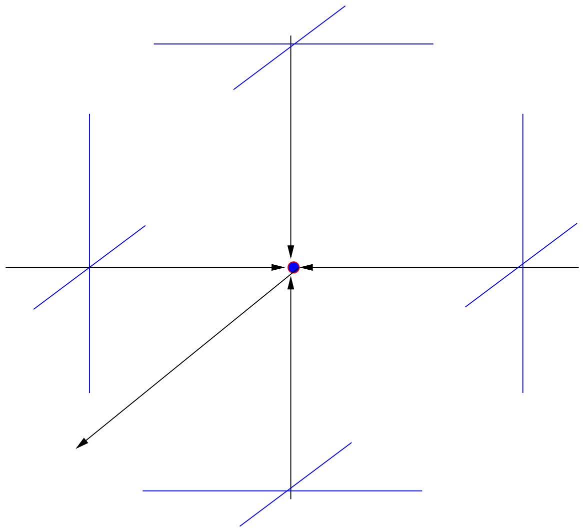


$\longrightarrow$ the other side only adds to e motion in sky plane

$\longrightarrow$ also linear polarization!

Q: what if isotropic initial radiation field?

isotropic initial radiation field:

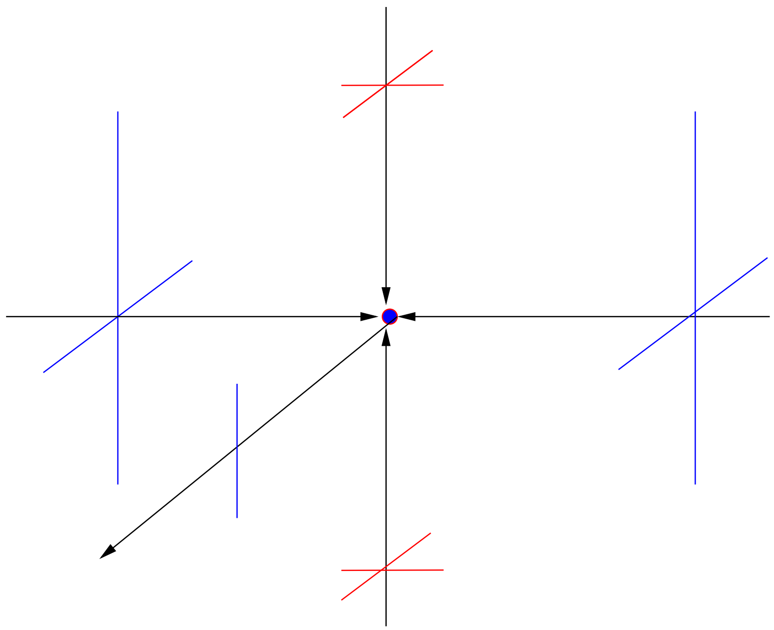


e motions in x and y sky directions cancel
$\longrightarrow$ no net polarization

Q: what initial radiation has quadrupole pattern?
i.e., less intense along one axis?

Q: lesson?

if initial radiation field has quadrupole intensity pattern



linear polarization!

lesson: polarization arises from Thomson scattering
when electrons "see" quadrupole anisotropies in radiation field

Awesomest Example of Thompson Polarization: the CMB

The CMB is nearly isotropic radiation field
arises from $\tau = 1$ "surface of last scattering" at $z = 1000$
when free e and protons "re" combined $ep \to H$

- *before recombination:*
  Thomson scattering of CMB photons, Universe opaque
- after recombination: no free e, Universe transparent

consider electron during last scatterings
sees and anisotropic thermal radiation field

consider point at hot/cold "wall"
locally sees *dipole* T anisotropy
net polarization towards us: zero! *Q: why?*

Q: what about edge of circular hot spot? cold spot?

polarization tangential (ring) around hot spots
radial (spokes) around cold spots
(superpose to "+" = zero net polarization–check!)

`www: WMAP polarization observations of hot and cold spots`

Note: polarization & T anisotropies *linked*
$\rightarrow$ consistency test for CMB theory and hence hot big bang

Polarization Observed

First detection: pre-WMAP!
⋆ DASI (2002) ground-based interferometer
at level predicted based on T anisotropies! Woo hoo!

WMAP (2003): first polarization-T correlation function

Planck (March 2013): much more sensitive to polarization
maybe a signature of inflation-generated gravitational radiation?