Astro 501: Radiative Processes
Lecture 17
October 5, 2018

Announcements:

- **Problem Set 5** due now
- good news: no Problem Set due next week
  bad news **Miderm Exam in class next Friday**

Midterm Exam

Time In class **Friday October 12**.
You will have the usual class time, 50 mintues

Topics
Everything up to and including Thomson scattering.
All material in Lectures 1–16 and Problem Sets 1-5 is fair game.

What to Bring
a pencil–or better, two pencils, and a calculator if you wish.
You may bring notes of any kind, but the exam is *closed book.*

Question Format
The homework questions point to important topics and questions. Given the exam time constriants, the problems will enerally be less involved than in homework, but rather the questions will emphasize an understanding of how to apply and interpret the tools we have developed.

Build Your Toolbox: Thomson Scattering

microphysics: matter-radiation interactions

Q: physical origin of Thomson scattering?

Q: physical nature of sources?

Q: spectrum characteristics?

Q: frequency range?

real/expected astrophysical sources of Thomson scattering

Q: where do we expect this to be important?

Q: relevant EM bands? temperatures?

Toolbox: Thomson Scattering

emission physics

- physical origin: scattering by non-relativistic free electrons
- physical sources: need free e^- $\rightarrow$ ionized gas
  scattering $\rightarrow$ photons conserved, need incident radiation
  scattering induces polarization even for unpolarized sources
- spectrum: Thomson scattered energy unchanged = coherent scattering
  σ_T indept of ν: spectral shape preserved in scattered radiation

astrophysical sources of Thomson scattering

- sites are illuminated and highly ionized gas: stellar interiors, stellar coronae, hot nebulae (Hii regions), early Universe
- EM bands radio to X-ray
  for γ-rays relativistic effects are important $\rightarrow$ Compton
- temperatures up to $\sim 10^6$ K
  above this, relativistic effects are important $\rightarrow$ Compton

Bremsstrahlung

Bremsstrahlung

German lesson for today:

Bremse = brake (as in stopping)

Strahlung = radiation

$\rightarrow$ Bremsstrahlung = "breaking radiation"

= radiation from decelerated charge particles

Consider a **dilute plasma** at temperature T, with

- free ions: charge $+Ze$, number density $n_{\rm i}$
- free electrons: charge $-e$, number density n_e

Q: astrophysical examples? `www: awesome example`

Q: what microphysics what will cause the plasma to emit?

i.e., what interactions will occur?

Q: which particles will radiate more?

dilute plasma = low particle density = typical in astrophysics
$\longrightarrow$ three-body collisions unlikely; ignore these
$\longrightarrow$ focus on two-body collisions

possible interactions: Coulomb forces between particle pairs

- electron-electron
- ion-ion
- electron-ion

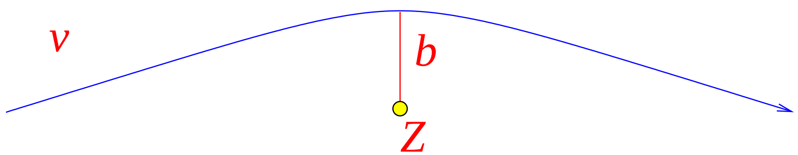
But electrons repel each other!
don't approach closely: electron-electron acceleration weak

electron and ion attracted and scattered by same Coulomb force
But $a_{\rm i}/a_e = m_e/m_{\rm i} < 10^{-3}$ $\rightarrow$ ion acceleration negligible
$\longrightarrow$ focus electron acceleration in static field of ion
electron-ion radiation dominates

Order of Magnitude Expectations

start with *classical, nonrelativistic* picture

consider a free, unbound electron with asymptotic speed v moving in Coulomb field of stationary ion



let b = *the distance of closest approach* or **impact parameter**

Q: estimate of maximum acceleration?

Q: duration of acceleration? velocity change? radiation frequency

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Recall the *Spirit of Order-of-Magnitude*:

- ignore all dimensionless constants, e.g., "small circle approximation" $2\pi \approx 1$
- lower expectations for precision
- use rough result to guide more careful calculations

maximum acceleration: Coulomb acceleration at closest approach

$$a_{\text{max}} \sim \frac{Ze^2}{m_e b^2} \tag{1}$$

duration of acceleration: **collision time**

$$\tau \sim \frac{b}{v} \tag{2}$$

velocity change

$$\Delta v \sim a_{\text{max}} \, \tau \sim \frac{Ze^2}{m_e b v} \sim \left(\frac{Ze^2/b}{m_e v^2} \right) v \tag{3}$$

frequency of radiation: use only timescale in problem

$$\omega \sim \frac{1}{\tau} \sim \frac{v}{b} \tag{4}$$

Q: what is maximum radiated power? radiated energy? energy per unit freq?

maximum radiated power is

$$P_{\mathrm{max}} \sim \frac{e^2 a_{\mathrm{max}}^2}{c^3} \sim \frac{e^2 \Delta v^2}{c^3 \ \tau^2} \sim \frac{Z^2 e^6}{m_e v^2 b^2 \ \tau^2} \tag{5}$$

radiated energy

$$\Delta W \sim P_{\mathrm{max}} \ \tau \sim \frac{Z^2 e^6}{m_e v^2 b^2 \ \tau} \tag{6}$$

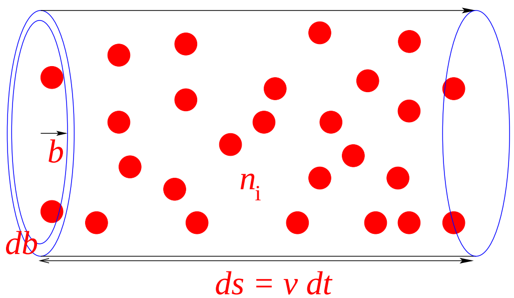
radiated energy per unit frequency

$$\frac{\Delta W}{\Delta \nu} \sim \frac{\Delta W}{\omega} \sim \frac{Z^2 e^6}{m_e v^2 b^2} \tag{7}$$

this energy radiated per electron-ion encounter at distance b

electron with speed v moves encounters ion number density n_{i}

- we want number of ions $d\mathcal{N}_{\mathrm{i}}$ that e encounters out to distance $\sim b$ in time dt *Q: which is?*
- *Q: what is typical rate of energy emitted per electron?*



in cylindrical distance $(b, b+db)$, volume swept is

$$dV = 2\pi\ b\ db\ ds = 2\pi\ v\ b\ db\ dt \tag{8}$$

i.e., $dV \sim b^2\ v\ dt$

thus number of ions encountered is

$$d\mathcal{N}_{\rm i} = n_{\rm i}\ dV\ \ \sim n_{\rm i}\ b^2\ v\ dt \tag{9}$$

Thus the rate of energy emitted = *power emitted per* e is

$$\frac{dP_{{\rm per}\,e}}{d\nu} = \frac{\Delta W}{\Delta \nu}\frac{d\mathcal{N}_{\rm i}}{dt} \sim \frac{e^6 Z^2}{m_e c^3 v} n_{\rm i} \tag{10}$$

Q: and so what is emission coefficient j_ν*?*

Our order-of-magnitude estimate for the emission coefficient from nonrelativistic bremsstrahlung:

$$j_\nu = n_e \frac{dP_{\mathrm{per}\,e}}{d\nu} \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_{\mathrm{i}} \tag{11}$$

Q: what's the basic physical picture?

Q: notable features? what didn't we get from order of mag?

Q: how can we do the classical calculation more carefully?

Bremsstrahlung: Physical Picture

we are interested in the motion of an electron through a plasma

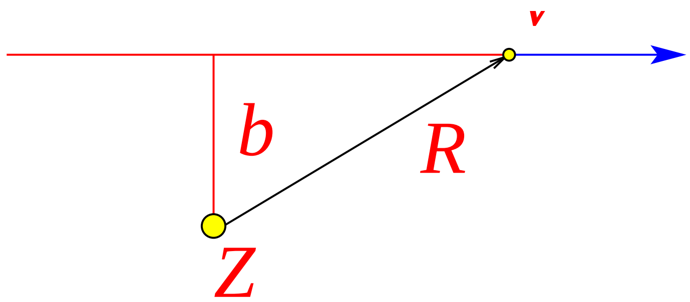
we approximate this as a series of

- *two-body electron-ion* scattering events
- *unbound Coulomb* trajectories: *hyperbolæ*
  $\rightarrow$ asymptotically free, scattered through small angle
- acceleration maximum at closest approach b
  lasting for scattering time $\tau = b/v$
- burst of radiation over this time, frequency $\nu \sim 1/\tau$

So net effect is

- many scattering events
- a series of small-angle scatterings
- and radiation bursts at different frequencies

Bremsstrahlung: Classical Calculation



Consider electron with initial speed v
with *impact parameter* b
moving fast enough so that
scattered through small angle

Strategy:

- treat as dipole with moment $\vec{d} = -e\vec{R}$
- take Fourier transform to find freq dependence $d(\omega)$
- find energy spectrum of radiation burst at b
- detailed derivation in Extras

Key insight: role of collision timescale $\tau = b/v$
velocity perturbation mode $v_\omega e^{i\omega t}$ response differs
for $\omega\tau \gg 1$ and $\omega\tau \ll 1$ *Q: how? and so?*

Result of procedure: energy emitted per electron at b

$$\frac{dW(b)}{d\omega} = \begin{cases} \frac{8Z^2e^2}{3\pi c^3 M_e^2 v b^2} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{12}$$

power emitted power per volume

$$\frac{dW(b)}{dV\ d\omega\ dt} = n_e \frac{dW}{d\omega}\frac{d\mathcal{N}_{\rm i}}{dt} = 2\pi n_e n_{\rm i} \int_{b_{\rm min}}^{b_{\rm max}} \frac{dW(b)}{d\omega}\ b\ db \tag{13}$$

approximate with low-frequency result:

$$q_\nu = 4\pi j_\nu = \frac{dW}{dV\ d\omega\ dt} = \frac{16Z^2e^2}{3\pi c^3 m_e^2 v} n_e n_{\rm i}\ \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \tag{14}$$

compare/contrast with order-of-magnitude:

- linear scaling with e and ion density
- $1/v$ scaling
- independence of b range $\rightarrow$ log dependence
- independence with ν, ω: "flat" emission spectrum

Impact Parameter Range

bremsstrahlung emission at speed v, frequency ω
depends *logarithmically* on the limits
$b_{\rm min}, b_{\rm max}$ of impact parameter
within our classical, small-angle-scattering treatment

lower limit

- quantum mechanics: $\Delta x \ \Delta p \gtrsim \hbar$
  $\rightarrow b_{\rm min}^{(1)} > h/mv$
- small-angle: $\Delta v/v \sim Ze^2/bmv^2 < 1$
  $\rightarrow b_{\rm min}^{(2)} > Ze^2/mv^2$

upper limit

for a fixed ω and v, max impact parameter is $b_{\rm max} \sim v/\omega$

fortunately: log dependence on limits
$\rightarrow$ results not very sensitive to details of choices

Single-Velocity Bremsstrahlung

convenient, conventional form for bremsstrahlung emission
also known as **free-free** emission

$$4\pi\ j_\omega(\omega, v) = \frac{16\pi}{3\sqrt{3}} \frac{Z^2 e^6}{m_e^2 c^3 v}\ n_i n_e\ g_{\rm ff}(\omega, v) \qquad (15)$$

eeq

uses the dimensionless correction factor or **Gaunt factor**

$$g_{\rm ff}(\omega, v) = \frac{\sqrt{3}}{\pi}\ \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \qquad (16)$$

- accounts for log factor
- typically $g_{\rm ff} \sim 1$ to few
- tables and plots available

Thermal Bremsstrahlung

so far: calculated bremsstrahlung emission for
a *single electron velocity* v
$\longrightarrow$ a "beam" of mono-energetic electrons

but in real astrophysical applications
there is a *distribution* of electron velocities
usually: a *thermal* distribution

so we wish to find
the *mean* or *expected* emission $\langle j_{\nu,\text{brem}} \rangle$
for a thermal distribution of velocities

Q: order-of-magnitude expectation?

Thermal Bremsstrahlung: Order-of-Magnitude

order-of-magnitude emission for single v:

$$j_\nu \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_\mathrm{i} \tag{17}$$

i.e., $j_\nu \sim 1/v$

thus, thermal average

$$\langle j_\nu \rangle \sim \frac{e^6 Z^2}{m_e c^3 v_T} n_e n_\mathrm{i} \tag{18}$$

with v_T a typical thermal velocity

find v_T from equipartition: $m_e v_T^2 \sim kT \rightarrow v_T \sim \sqrt{kT/m_e}$

Q: how do we approach the honest, detailed calculation?

Q: yet more new formalism?

Thermal Particles: Non-Relativistic Limit

recall: semiclassically, particle behavior in *phase space* $(\vec{x}, \vec{p})$
described by *distribution function* f:

- Heisenberg: minimum phase-space "cell" size $dx \; dp = h$
- particle number $dN = g/h^3 \; f(\vec{x}, \vec{p}) \; d^3\vec{x} \; d^3\vec{p}$

a *dilute*=non-degenerate, *non-relativistic* particle species
of mass m at temperature T
has distribution function

$$f_{\text{therm}}(p) \propto e^{-p^2/2mT} \tag{19}$$

and thus has number density $n \propto \int e^{-p^2/2m_e T} d^3\vec{p} \propto \int e^{-m_e v^2/2kT} d^3\vec{v}$

Q: how to compute thermal averaged bremsstrahlung emission?

Bremsstrahlung emissivity depends on electron properties via

$$j_\nu(\nu, T) = \langle j_\nu(\nu, v) \rangle \propto \left\langle \frac{g_{\mathrm{ff}}(\nu, v)\ n_e}{v} \right\rangle \tag{20}$$

where

$$\left\langle \frac{g_{\mathrm{ff}}(\omega, v)\ n_e}{v} \right\rangle \sim \int_{v_{\mathrm{min}}}^{\infty} \frac{g_{\mathrm{ff}}(\omega, v)}{v}\ e^{-m_e v^2/2kT}\ d^3\vec{v} \tag{21}$$

Note lower limit v_{min} at fixed ν
$\rightarrow$ minimum electron velocity needed to radiate photon of energy ν

Q: what value should this have? effect on final result?

energy conservation: to make photon of frequency ν
electron needs kinetic energy $m_e v^2/2 > h\nu$, so

$$v_{\rm min} = \sqrt{\frac{2h\nu}{m_e}} \tag{22}$$

thus exponential factor has

$$e^{-\frac{m_e v^2}{2kT}} = e^{-\frac{m_e v_{\rm min}^2}{2kT}} \; e^{-\frac{m_e (v^2 - v_{\rm min}^2)}{2kT}} = e^{-\frac{h\nu}{kT}} \; e^{-\frac{m_e (v^2 - v_{\rm min}^2)}{2kT}}$$

→ overall factor $e^{-h\nu/kT}$ in thermal average
→ photon production thermally suppressed at $h\nu > kT$

thermal bremsstrahlung = "free-free" emission result:

$$4\pi j_{\nu,\rm ff}(T) = \frac{2^5 \pi \; Z^2 \; e^6}{3 \; m_e c^3} \left(\frac{2\pi}{3 m_e k T} \right)^{1/2} \; \bar{g}_{\rm ff}(\nu, T) \; e^{-h\nu/kT} \; n_e \; n_{\rm i} \tag{23}$$

with $\bar{g}_{\rm ff}(\nu, T)$ the *velocity-averaged thermal Gaunt factor*
Q: spectral shape for optically thin plasma? implications?
Q: integrated emission?

$$4\pi j_{\nu,\mathrm{ff}}(T) = \frac{2^5\pi\ Z^2\ e^6}{3\ m_e c^3}\left(\frac{2\pi}{3m_e kT}\right)^{1/2}\ \bar{g}_{\mathrm{ff}}(\nu,T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} \quad (24)$$

main frequency dependence is $j_\nu \propto e^{-h\nu/kT}$

$\rightarrow$ flat spectrum, cut off at $\nu \sim kT/h$

$\rightarrow$ can use to determine temperature of hot plasma (PS5)

integrated bremsstrahlung emission:

$$4\pi j_{\mathrm{ff}}(T) = 4\pi \int j_{\nu,\mathrm{ff}}(T)\ d\nu \quad (25)$$

$$= \frac{2^5\pi\ Z^2\ e^6}{3\ m_e c^3}\left(\frac{2\pi kT}{3m_e}\right)^{1/2}\ \bar{g}_{\mathrm{B}}(T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} \quad (26)$$

$$= 1.4 \times 10^{-27}\ \mathrm{erg\ s^{-1}\ cm^{-3}}\ \bar{g}_{\mathrm{B}}\ \left(\frac{T}{\mathrm{K}}\right)^{\frac{1}{2}}\ \left(\frac{n_e}{1\ \mathrm{cm}^{-3}}\right)\ \left(\frac{n_{\mathrm{i}}}{1\ \mathrm{cm}^{-3}}\right)$$

with $\bar{g}_{\mathrm{B}}(T) \sim 1.2 \pm 0.2$ a frequency-averaged Gaunt factor

Q: all of this was for emission–what about the thermal bremsstrahlung absorption coefficient?

Thermal Bremsstrahlung Absorption

for *thermal* system, *Kirchoff's law:* $S_\nu = B_\nu(T) = j_\nu / \alpha_\nu$

thus we have

$$\alpha_{\nu,\mathrm{ff}} = \frac{j_{\nu,\mathrm{ff}}}{B_\nu(T)} = \frac{4\ Z^2\ e^6}{3\ m_e hc}\left(\frac{2\pi}{3 m_e kT}\right)^{1/2} \bar{g}_{\mathrm{ff}}(\nu, T)\ \nu^{-3}\ \left(1 - e^{-h\nu/kT}\right) n_e\, n_{\mathrm{i}}$$

limits:

- $h\nu \gg kT$: $\alpha_{\nu,\mathrm{ff}} \propto \nu^{-3}$
- $h\nu \ll kT$: $\alpha_{\nu,\mathrm{ff}} \propto \nu^{-2}$

Q: sketch optical depth vs ν? implications?

Bremsstrahlung Self-Absorption

bremsstrahlung optical depth at small ν:

$$\tau_\nu \propto \alpha_{\nu,\mathrm{ff}} \propto \nu^{-3} \tag{27}$$

guaranteed optically thick below some ν

$\longrightarrow$ free-free emission is absorbed inside plasma:

bremsstrahlung self-absorption

thus observed plasma spectra should have three regimes

- small ν: $\tau_\nu \gg 1$, optically thick, $I_\nu \to B_\nu \propto \nu^3$
- $h\nu < kT$: optically thin, $I_\nu \to j_\nu s$ *flat* vs ν
- $h\nu \gg kT$: thermally suppressed, $I_\nu \to j_\nu s \sim e^{-h\nu/kT}$

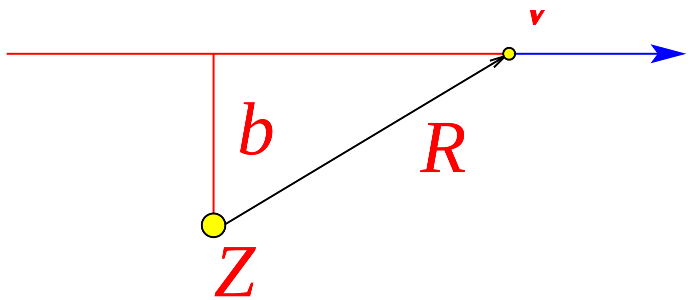
Q: expected X-ray count spectrum for supernova remnant?

`www: observations`

Director's Cut Extras

Bremsstrahlung: Classical Calculation

Consider electron with initial speed v
with *impact parameter* b
moving fast enough so that
scattered through small angle



dipole moment $\vec{d} = -e\vec{R}$, with second derivative

$$\ddot{\vec{d}} = -e\dot{\vec{v}} \tag{28}$$

take Fourier transform

$$-\omega^2\ \ddot{\vec{d}} = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t}\ dt \tag{29}$$

where: $\vec{v}(t)$ is *an unbound Coulomb trajectory*:
→ *hyperbola* in space, complicated function of time
but: $\dot{\vec{v}}(\omega)$ simplifies in limiting cases
→ compare ω and collision time $\tau = b/v$
Q: $\omega\tau \gg 1$? $\omega\tau \ll 1$?

$$-\omega^2 \ddot{\vec{d}} = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t}\, dt \tag{30}$$

but $\vec{v}(t)$ only changes on timescale τ:
for $\omega\tau \gg 1$, many oscillations during acceleration
complex phase averages out: $\vec{v}(\omega) \rightarrow 0$

for $\omega\tau \ll 1$, complex exponent unchanged during accel
phase unimportant: $\vec{v}(\omega) \rightarrow \int \dot{\vec{v}}\, dt = \Delta\vec{v}$

and thus the dipole moment has

$$\vec{d}(\omega) \rightarrow \begin{cases} \frac{e}{2\pi\omega^2}\Delta\vec{v} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{31}$$

Energy emitted per unit frequency

$$\frac{dW}{d\omega} \rightarrow \begin{cases} \frac{2e^2}{3\pi c^3}|\Delta\vec{v}|^2 & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{32}$$

Now find $\Delta\vec{v}$: for small deflection

$$\Delta v \approx \Delta v_{\perp} = \int F_z \, dt \tag{33}$$

$$= \frac{Ze^2}{m_e}\int \frac{b}{(b^2 + v^2t^2)^{3/2}}dt \tag{34}$$

$$= \frac{2Ze^2}{m_e b v} \tag{35}$$