Astro 501: Radiative Processes
Lecture 18
October 8, 2018

Announcements:

- good news: no Problem Set due this week
  bad news **Miderm Exam in class Friday**

Midterm Exam

Time In class **Friday October 12**.
You will have the usual class time, 50 minutes

Topics
Everything up to and including Thomson scattering.
All material in Lectures 1–16 and Problem Sets 1-5 is fair game.

What to Bring
a pencil–or better, two pencils, and a calculator if you wish.
You may bring notes of any kind, but the exam is *closed book.*

Question Format
The homework questions point to important topics and questions. Given the exam time constraints, the problems will enerally be less involved than in homework, but rather the questions will emphasize an understanding of how to apply and interpret the tools we have developed.

Last Time: Bremsstrahlung

Q: what is it?

bremsstrahlung also know as **free-free emission**
Q: what does this refer to?

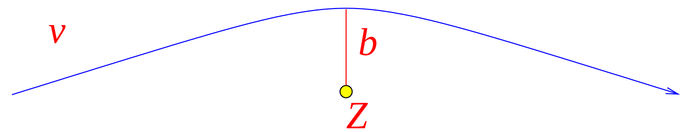
Q: what's the basic physical picture?

Q: what interactions, trajectories are relevant?

Q: what does bremsstrahlung emission j_ν depend on?

Bremsstrahlung = Free-Free Emission: Physical Picture

motion of *free* electrons
through a plasma of (*free*) ions



we approximate this as a series of

- *two-body electron-ion* scattering events
- *unbound Coulomb* trajectories: *hyperbolæ*
  $\rightarrow$ asymptotically free, scattered through small angle
- acceleration maximum at closest approach b
  lasting for scattering time $\tau = b/v$
- burst of radiation over this time, frequency $\nu \sim 1/\tau$

So net effect is

- many scattering events
- a series of small-angle scatterings
- and radiation bursts at different frequencies

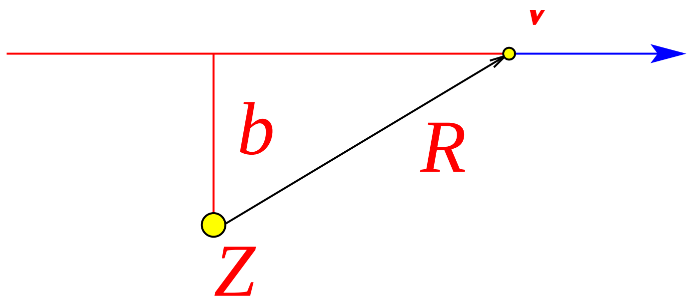
Our order-of-magnitude estimate for the emission coefficient from nonrelativistic bremsstrahlung:

$$j_\nu = n_e \frac{dP_{\mathrm{per}\,e}}{d\nu} \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_{\mathrm{i}} \tag{1}$$

Q: notable features? what didn't we get from order of mag?

Q: how can we do the classical calculation more carefully?

Bremsstrahlung: Classical Calculation



Consider electron with initial speed v
with *impact parameter* b
moving fast enough so that
scattered through small angle

dipole moment $\vec{d} = -e\vec{R}$, with second derivative

$$\ddot{\vec{d}} = -e\dot{\vec{v}} \tag{2}$$

take Fourier transform

$$-\omega^2 \ddot{\vec{d}} = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}}(t)\ e^{i\omega t}\ dt \tag{3}$$

where: $\vec{v}(t)$ is *an unbound Coulomb trajectory*:
$\rightarrow$ *hyperbola* in space, complicated function of time
but: $\dot{\vec{v}}(\omega)$ simplifies in limiting cases
$\rightarrow$ compare ω and collision time $\tau = b/v$
Q: $\omega\tau \gg 1$? $\omega\tau \ll 1$?

$$-\,\omega^2\ \ddot{\vec{d}} = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t}\ dt \tag{4}$$

but $\vec{v}(t)$ only changes on timescale τ:
for $\omega\tau \gg 1$, many oscillations during acceleration
complex phase averages out: $\vec{v}(\omega) \to 0$

for $\omega\tau \ll 1$, complex exponent unchanged during accel
phase unimportant: $\vec{v}(\omega) \to \int \dot{\vec{v}}\ dt = \Delta\vec{v}$

and thus only frequencies $\omega \lesssim \tau^{-1} = b/v$ contribute
and the dipole moment has

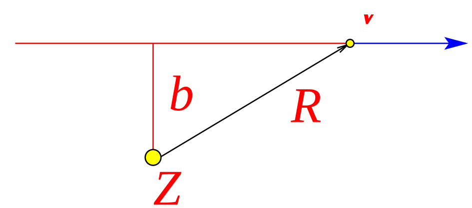
$$\vec{d}(\omega) \to \begin{cases} \frac{e}{2\pi\omega^2}\Delta\vec{v} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{5}$$

Energy emitted per unit frequency at impact parameter b:

$$\frac{dW}{d\omega} = \frac{8\pi e^2 \omega^4 \,|d(\omega)|^2}{3c^3} \rightarrow \begin{cases} \frac{2e^2}{3\pi c^3} |\Delta \vec{v}|^2 & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{6}$$

Now find $\Delta\vec{v}$: for small deflection

$$\begin{aligned} \Delta v &\approx \Delta v_\perp = \int F_z \; dt & (7) \\ &= \frac{Ze^2}{m_e} \int \frac{b}{(b^2 + v^2 t^2)^{3/2}} dt & (8) \\ &= \frac{2Ze^2}{m_e b v} & (9) \end{aligned}$$

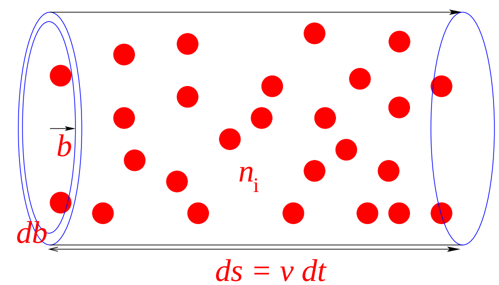


∞

Dipole formula give energy emitted per electron scattering at b

$$\frac{dW(b)}{d\omega} = \begin{cases} \frac{8Z^2e^2}{3\pi c^3 m_e^2 v b^2} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{10}$$

to include all impact parameters b:
weight by *collision rate*
$d\mathcal{N}_i/dt = 2\pi n_i v \; b \, db$ per electron
gives power emitted power per volume



$$\frac{dW(b)}{dV \; d\omega \; dt} = n_e \frac{dW}{d\omega}\frac{d\mathcal{N}_\mathrm{i}}{dt} = 2\pi n_e n_\mathrm{i} \int_{b_\mathrm{min}}^{b_\mathrm{max}} \frac{dW(b)}{d\omega} \; b \; db \tag{11}$$

Q: what will be b dependence?

using low-frequency result:

$$q_\nu = 4\pi j_\nu = \frac{dW}{dV\ d\omega\ dt} = \frac{16 Z^2 e^6}{3\pi c^3 m_e^2 v} n_e n_{\mathrm{i}}\ \ln\left(\frac{b_{\mathrm{max}}}{b_{\mathrm{min}}}\right) \tag{12}$$

compare/contrast with order-of-magnitude:

- $j_\nu \propto n_e n_i$: linear scaling with e and ion density
- $j_\nu \propto 1/v$ scaling
- independence of b range $\rightarrow$ log dependence
- independence with ν, ω: "flat" emission spectrum

Impact Parameter Range

bremsstrahlung emission at speed v, frequency ω
depends *logarithmically* on the limits
$b_{\mathrm{min}}, b_{\mathrm{max}}$ of impact parameter
within our classical, small-angle-scattering treatment

lower limit

- quantum mechanics: $\Delta x \; \Delta p \gtrsim \hbar$
  $\rightarrow b_{\mathrm{min}}^{(1)} > h/mv$
- small-angle: $\Delta v/v \sim Ze^2/bmv^2 < 1$
  $\rightarrow b_{\mathrm{min}}^{(2)} > Ze^2/mv^2$

upper limit

for a fixed ω and v, max impact parameter is $b_{\mathrm{max}} \sim v/\omega$

fortunately: log dependence on limits
$\rightarrow$ results not very sensitive to details of choices

Single-Velocity Bremsstrahlung

convenient, conventional form for bremsstrahlung emission also known as **free-free** emission

$$4\pi \; j_\omega(\omega, v) = \frac{16\pi}{3\sqrt{3}} \frac{Z^2 e^6}{m_e^2 c^3 v} \; n_\mathrm{i} n_e \; g_\mathrm{ff}(\omega, v) \tag{13}$$

eeq

uses the dimensionless correction factor or **Gaunt factor**

$$g_\mathrm{ff}(\omega, v) = \frac{\sqrt{3}}{\pi} \; \ln\left(\frac{b_\mathrm{max}}{b_\mathrm{min}}\right) \tag{14}$$

- accounts for log factor
- typically $g_\mathrm{ff} \sim 1$ to few
- tables and plots available

`www: awesome astrophysical example`

Q: how does this differ from our treatment?

Thermal Bremsstrahlung

so far: calculated bremsstrahlung emission for
a *single electron velocity* v
$\longrightarrow$ a "beam" of mono-energetic electrons

but in real astrophysical applications
there is a *distribution* of electron velocities
usually: a *thermal* distribution

so we wish to find
the *mean* or *expected* emission $\langle j_{\nu,\mathrm{brem}} \rangle$
for a thermal distribution of velocities

Q: order-of-magnitude expectation?

Thermal Bremsstrahlung: Order-of-Magnitude

order-of-magnitude emission for single v:

$$j_\nu \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_\mathrm{i} \tag{15}$$

i.e., $j_\nu \sim 1/v$

thus, thermal average

$$\langle j_\nu \rangle \sim \frac{Z^2 e^6}{m_e c^3 v_T} n_e n_\mathrm{i} \tag{16}$$

with v_T a typical thermal velocity

find v_T from equipartition: $m_e v_T^2 \sim kT \to v_T \sim \sqrt{kT/m_e}$

Q: how do we approach the honest, detailed calculation?

Q: yet more new formalism?

Thermal Particles: Non-Relativistic Limit

recall: semiclassically, particle behavior in *phase space* $(\vec{x}, \vec{p})$
described by *distribution function* f:

- Heisenberg: minimum phase-space "cell" size $dx\ dp = h$
- particle number $dN = g/h^3\ f(\vec{x}, \vec{p})\ d^3\vec{x}\ d^3\vec{p}$

a *dilute*=non-degenerate, *non-relativistic* particle species
of mass m at temperature T
has distribution function

$$f_{\text{therm}}(p) \propto e^{-p^2/2mT} \tag{17}$$

and thus has number density $n \propto \int e^{-p^2/2m_eT} d^3\vec{p} \propto \int e^{-m_ev^2/2kT} d^3\vec{v}$

Q: how to compute thermal averaged bremsstrahlung emission?

Bremsstrahlung emissivity depends on electron properties via

$$j_\nu(\nu,T) = \langle j_\nu(\nu,v) \rangle \propto \left\langle \frac{g_{\rm ff}(\nu,v)\ n_e}{v} \right\rangle \tag{18}$$

where

$$\left\langle \frac{g_{\rm ff}(\omega,v)\ n_e}{v} \right\rangle \sim \int_{v_{\rm min}}^{\infty} \frac{g_{\rm ff}(\omega,v)}{v}\ e^{-m_e v^2/2kT}\ d^3\vec{v} \tag{19}$$

Note lower limit $v_{\rm min}$ at fixed ν
$\rightarrow$ minimum electron velocity needed to radiate photon of energy ν

Q: what value should this have? effect on final result?

energy conservation: to make photon of frequency ν
electron needs kinetic energy $m_e v^2/2 > h\nu$, so

$$v_{\rm min} = \sqrt{\frac{2h\nu}{m_e}} \tag{20}$$

thus exponential factor has

$$e^{-\frac{m_e v^2}{2kT}} = e^{-\frac{m_e v_{\rm min}^2}{2kT}} \; e^{-\frac{m_e (v^2 - v_{\rm min}^2)}{2kT}} = e^{-\frac{h\nu}{kT}} \; e^{-\frac{m_e (v^2 - v_{\rm min}^2)}{2kT}}$$

$\rightarrow$ overall factor $e^{-h\nu/kT}$ in thermal average
$\rightarrow$ photon production thermally suppressed at $h\nu > kT$

thermal bremsstrahlung = "free-free" emission result:

$$4\pi j_{\nu,{\rm ff}}(T) = \frac{2^5 \pi \; Z^2 \; e^6}{3 \; m_e c^3} \left(\frac{2\pi}{3 m_e kT} \right)^{1/2} \; \bar{g}_{\rm ff}(\nu, T) \; e^{-h\nu/kT} \; n_e \; n_{\rm i} \tag{21}$$

with $\bar{g}_{\rm ff}(\nu, T)$ the *velocity-averaged thermal Gaunt factor*
Q: spectral shape for optically thin plasma? implications?
Q: integrated emission?

$$4\pi j_{\nu,\mathrm{ff}}(T) = \frac{2^5\pi\ Z^2\ e^6}{3\ m_e c^3}\left(\frac{2\pi}{3m_e kT}\right)^{1/2}\ \bar{g}_{\mathrm{ff}}(\nu,T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} \quad (22)$$

main frequency dependence is $j_\nu \propto e^{-h\nu/kT}$
$\rightarrow$ **flat spectrum**, cut off at $\nu \sim kT/h$
$\rightarrow$ can use to determine temperature of hot plasma (PS6)

integrated bremsstrahlung emission:

$$\begin{aligned} 4\pi j_{\mathrm{ff}}(T) &= 4\pi \int j_{\nu,\mathrm{ff}}(T)\ d\nu & (23)\\ &= \frac{2^5\pi\ Z^2\ e^6}{3\ h m_e c^3}\left(\frac{2\pi kT}{3m_e}\right)^{1/2}\ \bar{g}_{\mathrm{B}}(T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} & (24)\\ &= 1.4\times 10^{-27}\ \mathrm{erg\ s^{-1}\ cm^{-3}}\ \bar{g}_{\mathrm{B}}\ \left(\frac{T}{\mathrm{K}}\right)^{\frac{1}{2}}\ \left(\frac{n_e}{1\ \mathrm{cm}^{-3}}\right)\ \left(\frac{n_{\mathrm{i}}}{1\ \mathrm{cm}^{-3}}\right) \end{aligned}$$

with $\bar{g}_{\mathrm{B}}(T) \sim 1.2 \pm 0.2$ a frequency-averaged Gaunt factor
Q: all of this was for emission–what about the thermal bremsstrahlung absorption coefficient?