

Astro 501: Radiative Processes
Lecture 22
October 19, 2018

Announcements:

- **Problem Set 7** due now
- **Problem Set 6** due next Friday

Last time: thermodynamics of atomic states
Q: ratio of $2p$ to $1s$ states in hydrogen at T ?

Thermodynamics of Ionization

consider a hydrogen gas in thermodynamic equilibrium at T
ionization and recombination both occur



and the number densities n_e , n_p , and n_{H} adjust themselves
until the recombination and ionization rates are equal

this equilibrium determines a relationship among the densities
which we want to find

Method I (R&L):

starting point—the ratio of free electrons at speed v
to neutral hydrogen atoms in the ground state

$$\frac{\delta n_+(v)}{n_{\text{H}}} = \frac{\delta g(v)}{g_{\text{H}}} e^{-[E_e(v) - E_1]/kT} = \frac{\delta g(v)}{g_{\text{H}}} e^{-(B + m_e v^2/2)/kT} \quad (2)$$

where $B = -E_1$ is hydrogen binding energy

Boltzmann gives

$$\frac{\delta n_+(v)}{n_H} = \frac{\delta g(v)}{g_H} e^{-(B+m_e v^2/2)/kT} \quad (3)$$

and with statistical weight

$$g(v) = g_p g_e \quad (4)$$

$$= 2g_p \frac{dx dy dz dp_x dp_y dp_z}{h^3} \quad (5)$$

where electron volume element chosen so that number density $n_e = 1/d^3\vec{x} = 1/dx dy dz$, and thus

$$\frac{n_p}{n_H} = \frac{4\pi}{h^3 n_e g_H} g_p \int e^{-(B+p_e^2/2m)/kT} p^2 dp \quad (6)$$

$$= \frac{4\pi}{n_e g_H} \left(\frac{2kT}{m_e h^2}\right)^{3/2} e^{-B/kT} \int_0^\infty e^{-x^2} x^2 dx \quad (7)$$

and we arrive at the **Saha equation**

$$\frac{n_e n_p}{n_H} = \frac{g_e g_p}{g_H} \left(2\pi \frac{m_e m_p kT}{m_H h^2} \right)^{3/2} e^{-B_H/kT} \quad (8)$$

where *hydrogen binding energy*

$$B_H = (m_e + m_p - m_H)c^2 = 13.6 \text{ eV}$$

Q: behavior at high T? low T? does this make sense?

The Saha Equation

define *ionization fraction*

$$x_e = \frac{n_e}{n_{\text{tot}}} \quad (9)$$

with total electron number density $n_{\text{tot}} = n_e + n_{\text{H}}$
using $n_e = n_p$ (charge neutrality):

$$\frac{x_e^2}{1 - x_e} \approx \frac{2(2\pi m_e kT/h^2)^{3/2}}{n_{\text{tot}}} e^{-B_{\text{H}}/kT} = \frac{n_{\text{Q},e}}{n_{\text{tot}}} e^{-B_{\text{H}}/kT} \quad (10)$$

for $kT \gg B_{\text{H}}$, $x_e \rightarrow 1$: (nearly) fully ionized

for $kT \ll B_{\text{H}}$, $x_e \ll 1$: (nearly) fully neutral

but note that, e.g., temperature at which $x_e = 1/2$

⁵ also depends on particle density n_{tot}

Awesome Saha Example: Cosmic Recombination

the early universe: *hot!*

$kT \gg B_H \rightarrow$ totally ionized, $x_e \rightarrow 1$

present-day universe: on average, *cold!*

$T = 2.725$ K \rightarrow if no stars, U would be neutral, $x_e \rightarrow 0$

thus there was a transition: **(re)combination**

our mission: estimate $T_{\text{rec}} =$ when cosmic $x_e = 1/2$

Q: naïve, zeroth order estimate?

Q: how to improve?

naïvely, expect recombination when $kT_{\text{rec}} \sim B_{\text{H}}$
with $B_{\text{H}} = 13.6$ eV, this gives
 $T_{\text{rec,naive}} = B_{\text{H}}/k \sim 120,000$ K

but we can do better using Saha
exponential dependence on B_{H} , but also
dependence on n_{tot}

big-bang nucleosynthesis teaches* us that
the cosmic baryon-to-photon ratio is

$$\eta \equiv \frac{n_{\text{b}}}{n_{\gamma}} = 6 \times 10^{-10} \quad (11)$$

most baryons are hydrogen, so $n_{\text{tot}} \sim n_{\text{b}}$
and thus there are *many photons for each p and e*

Q: anticipated effect on T_{rec} ? higher or lower than $T_{\text{rec,naive}}$?

∟

*How? find out next semester in Nuclear and Particle Astrophysics!

many photons per p and $e \rightarrow$ very easy to ionize H

- when $kT < B_H$, there are still many photons in Wien tail with $h\nu > B_H$
- thus expect $T_{\text{rec}} < T_{\text{rec,naive}}$

in detail:

recall that $n_\gamma \sim (kT/hc)^3$, so

$$n_{\text{tot}} \sim \eta n_\gamma \sim \eta (kT/hc)^3 \quad (12)$$

and so Saha becomes

$$\frac{x_e^2}{1 - x_e} \sim \frac{1}{\eta} \left(\frac{m_e c^2}{kT} \right)^{3/2} e^{-B_H/kT} \quad (13)$$

note: $1/\eta \gg 1$ and $m_e c^2/kT \gg 1$

so when $x_e = 1/2$ we have (PS 7)

$$\infty \quad T_{\text{rec}} \simeq T_{\text{rec,naive}}/40 \sim 3000 \text{ K}$$

$$kT_{\text{rec}} \simeq 0.3 \text{ eV} \ll B_H$$

$$\text{and thus } 1 + z_{\text{rec}} = T_{\text{rec}}/T_0 \sim 1000$$

Saha Generalized

can generalize Saha to get ionization equilibrium
for any species having $a^+ + e \leftrightarrow a^0 + \gamma$

$$\frac{n_+ n_e}{n_0} = \frac{g_+ g_e}{g_0} \left(2\pi \frac{m_e m_+ kT}{m_0 h^2} \right)^{3/2} e^{-B/kT} \quad (14)$$

with B the binding energy

Radiative Transitions

Radiative Transitions

so far: thermal populations of bound states

now: *transitions* between states

leading to emission/absorption

we want a qualitative and quantitative understanding

qualitatively:

- what is the basic physics?
- *selection rules*: which transition are allowed?

quantitatively:

Q: what do we want to know?

quantitatively:

we want to describe the *strength* of transitions
in particular, the usual radiation transfer quantities

- emission coefficient j_ν
- absorption coefficient α_ν

these are closely related to Einstein coefficients

- A_{if} *spontaneous emission* rate per atom for $i \rightarrow f$
- B_{if} *stimulated emission* coefficient
- B_{fi} *true absorption* coefficient

recall: we found that, for $h\nu_{if} = E_i - E_f$

$$j_\nu = \frac{h\nu_{if} A_{if}}{4\pi} n_i \phi(\nu) \quad (15)$$

$$\alpha_\nu = \frac{h\nu_{if}}{4\pi} (B_{fi}n_f - B_{if}n_i) \phi(\nu) \quad (16)$$

$$(17)$$

with $\phi(\nu)$ the *line profile* function

Radiative Transitions: Eigenstates are Forever?

Recall the quantum mechanics of bound systems, e.g., H atom controlled by Hamiltonian operator $\hat{H} = \hat{p}^2/2m + V(\hat{r})$ with V the *potential* binding the constituents

wavefunctions are *energy eigenstates* $H\Psi_n = E_n\Psi_n$ with $\Psi_n(\vec{r}, t) = \psi_n(\vec{r})e^{-iE_n t/\hbar}$

Note:

- electron probability in space $dP/d\vec{r} = |\Psi(\vec{r})|^2$
for eigenstate: $dP/d\vec{r} = |\psi_n(\vec{r})|^2$ is *independent of time*
- expectation values within eigenstate also *time independent*
e.g., $\langle \vec{r} \rangle = \int |\psi(\vec{r})|^2 \vec{r} d^3\vec{r}$
- thus *expected dipole moment* $\langle \vec{d} \rangle = e \langle \vec{r} \rangle$
also *time independent*

Q: implications for eigenstate evolution? for radiation?

Q: Whaaa? how can it be that atoms radiate at all??

Spontaneous Emission: Dipole Approximation

for *unperturbed bound* system: once in *any* eigenstate

- stay there forever! including in excited states!
- no changing dipole moment (or change in any higher moments)
→ no radiation!

but if an external perturbation is present

wavefunction *mixes states*

Q: what could be the source of perturbation?

A Quantum Vacuum is Not Boring!

External perturbation needed to drive transitions in bound states
but luckily: one source is guaranteed!

the electromagnetic field is quantized!

- closely analogous to *simple harmonic oscillator*
where $H = p^2/2m + m\omega^2 x^2/2$ leads to
quantized states $E_n = (n + 1/2)\hbar\omega$
ground state $n = 0$: $\psi_0 \neq 0$! zero point energy $E_0 = \hbar\omega/2$!
- EM field Hamiltonian $H = (E^2 + B^2)/8\pi$
for experts, true analogy is potentials: $H \sim \dot{A}^2 + (\nabla A)^2 \sim \dot{A}_\omega^2 + \omega^2 A_\omega^2$
leads to quantized states, each with $E = \hbar\omega$
but also **zero point** fluctuations for $n = 0$ quanta!

Lessons:

- *EM vacuum: ground state ($n = 0$). Not empty!*
- vacuum fluctuations inevitable, always present
- perturbs bound systems, drives transitions

Director's Cut Extras

The Road to Saha: Method II—Chemical Equilibrium

recall that the number of states for a particle is related to its *distribution function* f via

$$dN = \frac{g}{h^3} f d^3x d^3p \quad (18)$$

where f counts states in phase space

i.e., *translational* degrees of freedom

and where g counts *internal* degrees of freedom

e.g., for a free electron, $g_e = 2s_e + 1 = 2$

a particle species in thermal (in fact, kinetic) equilibrium at T has

$$f = \frac{1}{e^{(E-\mu)/kT} \pm 1} \quad (19)$$

where $\pm \leftrightarrow$ fermion/boson

¹⁷ and $E(p) = \sqrt{(cp)^2 + (mc^2)^2} \stackrel{\text{nonrel}}{=} mc^2 + p^2/2m$

and where μ is the *chemical potential* (more on this soon)

distribution function, $\pm \leftrightarrow$ fermion/boson

$$f = \frac{1}{e^{(E-\mu)/kT} \pm 1} \quad (20)$$

for nonrelativistic, nondegenerate gasses of interest, $f \ll 1$
 $\rightarrow e^{(E-\mu)/kT} \gg 1$, and thus we get

Maxwell-Boltzmann, same for fermions and bosons

$$f \approx f_{\text{MB}} = e^{(mc^2-\mu)/kT} e^{-p^2/2mkT} \quad (21)$$

and thus *number density* is

$$n = \frac{g}{h^3} \int d^3p f = g \left(\frac{2\pi mkT}{h^2} \right)^{3/2} e^{(mc^2-\mu)/kT} \quad (22)$$

thus $n(T, \mu)$: density depends not only on T
but also on chemical potential(?)

nonrelativistic particle density

$$n(T, \mu) = \frac{g}{h^3} \int d^3p f = g \left(\frac{2\pi m k T}{h^2} \right)^{3/2} e^{(mc^2 - \mu)/kT} \quad (23)$$

if chemists invented μ , isn't it boring? Fair question, but no!

chemical potential μ : bad name, important quantity

consider a particle species with $\mu = 0$

Q: *effect on n if T changes?*

Q: *what would it be like if air in this room obeyed this rule?*

Q: *so what does it mean physically if $\mu = 0$?*

Q: *so what does it mean physically if $\mu \neq 0$?*

Chemical Potential

if $\mu = 0$:

density obeys $n(T) = g n_q(T) e^{-mc^2/kT}$

with the *quantum concentration* $n_q = (2\pi mkT/h^3)^{3/2}$

→ density is a fixed, universal function of T

→ all $\mu = 0$ gasses have *same density at same T !*

furthermore:

- since nonrel, $kT \ll mc^2 \rightarrow n$ small!

- but n is an increasing function of T

→ so in fixed volume, *raising T adds new particles!*

lesson: particles with $\mu = 0$ are *not conserved!*

20 in fact, we already saw a (relativistic) example: photons!
recall Planck dist func $f = 1/(e^{E/kT} - 1)$: boson with $\mu = 0$

we are usually interested in species that *are* conserved
e.g., protons, neutrons conserved due to baryon number
electrons conserved due to charge and lepton number

in that case: non-relativistic equilibrium density
determined not by temperature, but by conservation law

$$n_{\text{cons}} = g n_q e^{-(mc^2 - \mu)/kT} \quad (24)$$

this sets value of μ

Why is all of this useful?!

because in a reaction $a + b \leftrightarrow c + d$

the chemical potentials of each species
are related by

$$\mu_a + \mu_b = \mu_c + \mu_d \quad (25)$$

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Q: so what about the case $p + e \leftrightarrow \text{H} + \gamma$?

for $p + e \leftrightarrow H + \gamma$, we have

$$\mu_p + \mu_e = \mu_H \quad (26)$$

because $\mu_\gamma = 0$

using this and $n_i = g_i n_Q e^{-(m_i c^2 - \mu_i)/kT}$, we have
the **Saha equation**

$$\frac{n_e n_p}{n_H} = \frac{g_e g_p}{g_H} \left(2\pi \frac{m_e m_p kT}{m_H h^2} \right)^{3/2} e^{-B_H/kT} \quad (27)$$

where *hydrogen binding energy*

$$B_H = (m_e + m_p - m_H)c^2 = 13.6 \text{ eV}$$

Q: *behavior at high T? low T? does this make sense?*