

Astro 501: Radiative Processes
Lecture 23
October 22, 2018

Announcements:

- **Problem Set 7** due Friday
- seminars next semester

Last time:

the quantum mechanics of bound states

energy eigenstates *Q: time dependence? dipole moment?*

Q: what is required for spontaneous emission?

Feeling Perturbed

consider two eigenstates:

upper level E_u and *lower level* E_ℓ

when perturbed $V \rightarrow V + \delta V$, new wavefunction

$$\Psi = a_\ell e^{-iE_\ell t/\hbar} \psi_\ell + a_u e^{-iE_u t/\hbar} \psi_u \quad (1)$$

with nonzero amplitudes a_i

this changes expectations values

$$\langle \vec{d} \rangle = |a_\ell|^2 \langle \vec{d} \rangle_\ell + |a_u|^2 \langle \vec{d} \rangle_u + 2\text{Re} \left(a_\ell^* a_u e^{i\omega_{\ell u} t} \right) \quad (2)$$

creates *time changing dipole* and thus **radiation**

² at frequency $\omega_{\ell u} = (E_u - E_\ell)/\hbar!!$

Spontaneous Dipole Emission: Wild West Derivation

Full derivation: requires quantum electrodynamics
i.e., quantum treatment of electromagnetic field
Sketched in R&L and in Extras below
Here: cowgirl/cowboy “horseback” derivation

consider a transition from an *upper level* E_u
to a *lower level* E_ℓ
expected time-changing dipole component is

$$\vec{d} \equiv \langle \vec{d} \rangle \sim e e^{-i\omega_{lu}t} \langle u | \vec{r} | \ell \rangle \quad (3)$$

and so dipole acceleration is

$$\ddot{\vec{d}} \sim e \omega_{lu}^2 e^{-i\omega_{lu}t} \langle u | \vec{r} | \ell \rangle \quad (4)$$

ω

Q: and so?

in dipole approximation, **Larmor power per atom** is

$$P_{ul} = \frac{2|\dot{d}|^2}{3c^3} \sim \frac{\omega_{lu}^4}{c^3} |d_{ul}|^2 \quad (5)$$

- transition driven by dipole operator

$$\vec{d}_{ul} = e \int \psi^* \vec{r} \psi_u dV$$

between initial and final states

- zero when no dipole moment—*forbidden* transitions!
but higher multipole transitions may still go

now we are ready for Einstein A_{ul} ! Q: *how?*

Spontaneous Dipole Emission

the *power* emitted in $u \rightarrow \ell$ transition:

$$P_{ul} \sim \frac{\omega_{lu}^4}{c^3} |d_{ul}|^2 \quad (6)$$

energy released per transition $E_{ul} = \hbar\omega_{lu}$

so estimate transition rate per atom as

$$\frac{dN_{ul}}{dt} \sim \frac{P_{ul}}{E_{ul}} \sim \frac{\omega_{lu}^3}{\hbar c^3} |d_{ul}|^2 \quad (7)$$

exact Einstein coefficient for *spontaneous emission*

$$A_{ul} = \frac{64\pi^4 \nu_{ul}^3 |d_{ul}|^2}{3c^3 h} \quad (8)$$

5 Q: what about absorption and stimulated emission?

Einstein Coefficients

the Einstein coefficients in the electric dipole approximation are:

- *spontaneous emission*

$$A_{ul} = \frac{64\pi^4 \nu_{ul}^3 |d_{ul}|^2}{3c^3 h} = \frac{2\nu^3}{c^2 h} B_{lu} \quad (9)$$

- true *absorption*

$$B_{lu} = \frac{8\pi^2}{3c\hbar^2} |d_{lu}|^2 = \frac{32\pi^4}{3ch} |d_{lu}|^2 \quad (10)$$

for *non-degenerate atomic levels* with $g_l = g_u = 1$ we have

- *stimulated emission*

$$B_{ul} = B_{lu} \quad (11)$$

this gives (at least in principle) a direct means to connect

- the radiative coefficients j_ν and α_ν
to the atomic properties encoded in the dipole moment d_{ul}

recall that the absorption coefficient is

$$\alpha_\nu = \frac{h\nu}{4\pi} n_\ell B_{\ell u} \phi(\nu) \quad (12)$$

and so writing this in terms of the *absorption cross section* $\sigma_{\ell u}$

$$\alpha_\nu = n_\ell c \sigma_{\ell u}(\nu) \quad (13)$$

so that the cross section and Einstein coefficient are related by

$$\sigma_{\ell u}(\nu) = \frac{h\nu}{4\pi c} B_{\ell u} \phi(\nu) \quad (14)$$

integrating and using $\int \phi(\nu) d\nu = 1$, we have

$$B_{\ell u} = \frac{4\pi c}{h\nu_\ell} \int \sigma_{\ell u}(\nu) d\nu \quad (15)$$

and thus our expressions for $B_{\ell u}$ also give $\sigma_{\ell u}$

Oscillator Strength

If the electron moves as a *damped classical oscillator* with natural (resonant) frequency ω_0 then (PS7) absorption rate is $B_{lu}^{\text{classical}} J(\nu_{lu})$ with

$$B_{lu}^{\text{classical}} = \frac{4\pi^2 e^2}{h\nu_{lu} m_e c} \quad (16)$$

it is thus convenient write

$$B_{lu} \equiv f_{lu} B_{lu}^{\text{classical}} \quad (17)$$

$$\sigma_{lu}(\nu) = \frac{\pi e^2}{m_e c} f_{lu} \phi(\nu) \quad (18)$$

where the dimensionless **oscillator strength** is

$$f_{lu} = \frac{m_e}{\pi e^2} \int \sigma_{lu}(\nu) d\nu = \frac{2m_e}{3\hbar^2 g_{le} e^2} (E_u - E_l) \sum |d_{lu}|^2 \quad (19)$$

Q: what about f_{ul} ?

Einstein $g_{\ell}B_{\ell u} = g_u B_{u\ell}$, and since we have absorption

$$g_u f_{ul} = -g_{\ell} f_{\ell u} \quad (20)$$

so *emission oscillator strengths are negative*

if we sum over all transitions from $i \rightarrow j$,
can show that *one-electron atoms have*

$$\sum_{j \text{ final}} f_{ij} = 1 \quad (21)$$

where strong transitions have $f_{ij} \sim 1$
and *N-electron atoms have*

$$\sum_{j \text{ final}} f_{ij} = N \quad (22)$$

the *Thomas-Riche-Kuhn sum rule*

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Q: What if two states have no dipole moment: $d_{if} = 0$?

Beyond the Dipole

Our focus has been on electric dipole radiation where Larmor gives power $P_{E1} \sim \dot{d}^2$

but radiation also results from other time-changing charge multipoles

e.g., magnetic dipole $P_{M1} \sim \dot{m}^2$

electric quadrupole $P_{E2} \sim \ddot{Q}^2$

⇒ **higher multipoles can radiate when dipole forbidden ($d = 0$)!**

But there is a cost! for system of size a , freq ω :

- electric dipole $d \sim ea$, $P \sim \omega^2 e^2 a^2$
- magnetic dipole $m \sim Ia^2/c \sim ewa^2/c \sim d v/c$
- electric quadrupole $Q \sim ea^2$, $P \sim \omega^6 Q^2/c^2$

magnetic dipole and electric quadrupole power down by $\sim (v/c)^2$

dipole radiation dominates unless forbidden

Electric Dipole Selection Rules

When is a transition between states i and f possible?

in general: the transition probability is *always nonzero* but can be very small if the transition is suppressed, usually due to a symmetry
e.g., a forbidden dipole can have a nonzero quadrupole rate

focus on selection rules for *electric dipole transitions*
where recall that the dipole matrix element is

$$\vec{d}_{fi} = e \int \psi_f^* \sum_{\text{electrons } j} \vec{r}_j \psi_i d^3x \quad (23)$$

Laporte's rule:

no transitions between two states of the same parity

¶ Q: what is a parity transformation?

Q: why is $\vec{d}_{fi} = 0$ if i and j have same parity?

a parity transformation is the mapping $\vec{r} \rightarrow -\vec{r}$

note: electron wavefunctions are angular momentum eigenstates

and angular momentum eigenstates are parity eigenstates

thus: wavefunctions have definite parity

$$\psi_k(-\vec{r}) = \pi_k \psi_k(\vec{r}), \text{ with } \pi_k = \pm 1$$

thus if $\pi_i = \pi_f$, then

$$\vec{d}_{fi} \rightarrow \vec{d}'_{fi} = -e \int \psi_f^* \sum_j \vec{r}_j \psi_i d^3x = -\vec{d}_{fi} \quad (24)$$

and thus $\vec{d}_{fi} = 0$: *no transitions when parity unchanged*

the parity of an electron configuration (set of states)

is set by the electron angular momenta:

parity is $(-1)^{\sum \ell_i}$, where each electron has ℓ_i

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thus we conclude: *no transitions between the same configuration*

Note that the atomic wavefunction is really a function $\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$ over all N electron coordinates and at our level of approximation can be written in terms of single-electron wavefunctions $u_a(\vec{r}_1) u_b(\vec{r}_2) \dots u_k(\vec{r}_N)$ where $\int u_a^* u_a d^3x = 1$

thus the dipole operator \vec{r}_j picks out the wavefunctions for a *single electron*, involving $\int u_{a'}^* \vec{r}_j u_a d^3r_j$

Q: *implications?*

the dipole operator only involves $\int u_a^* \vec{r}_j u_a d^3 r_j$
for a *single electron*

thus we conclude

- all other electron wavefunctions remain the same
- *one electron jumps* per transition
- the transition dipole moment is that of the jumping electron
- in the jump the *parity change* is $(-1)^{\Delta\ell}$

vector nature of dipole operator imposes conditions on
single electron states in transitions:

$$\Delta\ell = \pm 1 \quad (25)$$

$$\Delta m = 0, \pm 1 \quad (26)$$

rules for *total angular momentum quantum numbers*

$$\Delta S = 0 \quad (27)$$

$$\Delta L = 0, \pm 1 \quad (28)$$

$$\Delta J = 0, \pm 1 \quad \text{except } J = 0 \text{ to } J = 0 \quad (29)$$

note that we can have $\Delta L = 0$

but always must have $\Delta \ell = \pm 1$

examples:

- $3s \ ^2S_{1/2} \rightarrow 4s \ ^2S_{1/2}$

$\Delta \ell = 0$: forbidden!

- $2p \ ^2P_{1/2} \rightarrow 3d \ ^2D_{5/2}$

$\Delta \ell = 1$, OK!

$\Delta L = 1$, OK!

$\Delta S = 0$, OK!

$\Delta J = 2$, forbidden!

Director's Cut Extras

The Semiclassical Approach

Deriving the general Einstein A and B coefficient for transitions between two atomic states from first principles is a big job

we will take a “first-ish” principles approach
sketch what goes into the final result

we will work in the *semiclassical limit*

- treat the atomic states quantum mechanically
- but treat the radiation classically

→ i.e., in the limit of large photon occupation f

⌈ good for getting Einstein B , bad for A Q : *why?*

Q : *but what's the workaround if we know B ?*

classical radiation \leftrightarrow large photon occupation f

absorption and stimulated emission: rate proportional to $\bar{J}_\nu = \int I_\nu d\Omega$

and recall $I_\nu = 2\nu^2/c^2 f$

\rightarrow so rate $\propto \int f d\Omega$ works even down to small f

spontaneous emission: involves single photons

correct analysis demands quantum treatment of radiation field

but luckily Einstein says: $A_{if} = (2h\nu_{if}^3/c^2)B_{fi}$

so if we find B , then use this to get A

thus: we will calculate *absorption*

So we will:

- treat atoms quantum mechanically, and
- treat radiation as a perturbation, in the form of an *external classical* EM field

Q: how do we describe formally the unperturbed system?

Q: how do we introduce the perturbation?

The Electromagnetic Hamiltonian

recall quantum mechanics: stationary atomic states $|n\rangle$ are governed by the time-independent Schrödinger equation

$$H_0 |n\rangle = E_n |n\rangle \quad (30)$$

in terms of wavefunctions $\psi_n(x) = \langle x|n\rangle$,

$$H_0 \psi_n = E_n \psi_n \quad (31)$$

with H_0 the **Hamiltonian** operator for the atom and includes the e -nucleus EM interactions and E_n is the energy of state n

add an external classical field with 4-potential (ϕ, \vec{A}) the **relativistic Hamiltonian** for an electron is

$$H = \sqrt{(c\vec{p} + e\vec{A})^2 + (m_e c^2)^2} - e\phi \quad (32)$$

for experts: gives right equation of motion in Hamilton's eqs
Q: *limit of no field? non-relativistic limit?*

The Relativistic Hamiltonian

full relativistic Hamiltonian for an electron

$$H = \sqrt{(c\vec{p} + e\vec{A})^2 + (m_e c^2)^2} - e\phi \quad (33)$$

non-relativistic limit: $cp \ll m_e c^2$

$$H = \frac{1}{2m_e} \left(\vec{p} + \frac{e\vec{A}}{c} \right)^2 - e\phi \quad (34)$$

$$= \frac{p^2}{2m_e} + \frac{e}{m_e c} \vec{A} \cdot \vec{p} + \frac{e^2 A^2}{2m_e c^2} - e\phi \quad (35)$$

plus a constant term $m_e c^2$ which we ignore *Q: why?*

note: we have used the “Coulomb gauge” for the perturbation

$$\nabla \cdot \vec{A} = 0 = \phi$$

we can write the non-relativistic Hamiltonian as

$$H = H_0 + H_1 + H_2 \quad (36)$$

where the *unperturbed atomic Hamiltonian* is H_0 ,
the perturbation *first order in A* is

$$H_1 = \frac{e}{m_e c} \vec{A} \cdot \vec{p} \quad (37)$$

and the perturbation *second order in A* is

$$H_2 = \frac{e^2 A^2}{2m_e c^2} \quad (38)$$

there is a beautiful physical interpretation of the terms:

- H_1 describes one-photon emission processes
- H_2 describes two-photon emission processes

Q: *relative importance of the two terms?*

order-of-magnitude estimate of the ratio of terms, in H atom:

$$\eta = \frac{H_1}{H_2} \sim \frac{epA/m_e c}{e^2 A^2/m_e c^2} \sim \frac{ev/c}{\alpha^2 a_0 A} \quad (39)$$

external electric field $E \sim 1/c \text{ partial}_t A \sim \nu/c A$
 and in H: $v/c \sim \alpha$, and $h\nu \sim e^2/a_0$ so $h\nu/c \sim \alpha/a_0$

$$\eta^2 \sim \frac{h\nu}{a_0^3 E^2} \quad (40)$$

but $E^2/h\nu \sim n_{\text{ph}}$, the photon density in the external field

$$\eta^2 \sim \frac{1}{n_{\text{ph}} a_0^3} \sim \left(\frac{10^{25} \text{ photons/cm}^3}{n_{\text{ph}}} \right) \quad (41)$$

at the Sun's surface $n_{\text{ph}} \sim 10^{12}/\text{cm}^3$

23 lesson: $\eta \gg 1$ for (almost) all astro applications
 → *ignore the two-photon term H_2*

The Transition Probability

we want the *probability* for transition $i \rightarrow f$
where the unperturbed wavefunctions satisfy $H_0 \psi_k = E_k \psi_k$ this
probability is *time-dependent*

the perturbing field generates nonzero amplitude for states $n \neq i$
so write time-dependent wavefunction as

$$\psi(t) = \sum_k a_k(t) \psi_k e^{-iE_k t/\hbar} \quad (42)$$

Q: $a_k(t)$ for system without perturbation? behavior with perturbation?

for at time-dependent potential, standard quantum mechanics gives

the probability P_{fi} to go from state $i \rightarrow f$

$$P_{fi} = w_{fi} t \quad (43)$$

with t the time the perturbation acts and the *transition probability per unit time*

$$w_{fi} = \frac{4\pi^2 |H(\omega_{fi})|^2}{\hbar^2 t} \quad (44)$$

where $H_{fi}(\omega) = (2\pi)^{-1} \int_0^t H_{fi}(t) e^{i\omega t'} dt'$
 with the *matrix element* $H_{fi} = \int \psi_f^* H_1 \psi_i d^3x$
 and where $\hbar\omega_{fi} = E_f - E_i$

if we have multiple atomic electrons, then perturbation is sum

$$H_1 = \frac{e}{m_e c} \sum_j \vec{A} \cdot \vec{p}_j = \frac{ie\hbar}{m_e c} \vec{A} \cdot \sum_j \nabla_j \quad (45)$$

let the perturbing field have:

- $\vec{A}(\vec{r}, t) = \vec{A}(t) e^{i\vec{k}\cdot\vec{r}}$, with
- $\vec{A}(t') = 0$ outside of $(0, t)$

then the Fourier transform of the matrix element is

$$H_{fi} = \vec{A}_{fi}(\omega_{fi}) \cdot \frac{ie\hbar}{c} \langle f | e^{i\vec{k}\cdot\vec{r}} \sum_j \nabla_j | i \rangle \quad (46)$$

where $\langle f | e^{i\vec{k}\cdot\vec{r}} \sum_j \nabla_j | i \rangle = \sum_j \int \psi_f^* \nabla_j \psi_i d^3x$ is *time-independent*

write $\vec{A} = A \mathbf{e}$ with unit polarization vector \mathbf{e} :

$$w_{fi} = \frac{4\pi^2 e^2}{m_e c^2 t} |A(\omega_{fi})|^2 \left| \langle f | e^{i\vec{k}\cdot\vec{r}} \mathbf{e} \cdot \sum_j \nabla_j | i \rangle \right|^2 \quad (47)$$

26 note that $w_{fi} \propto |A(\omega_{fi})|^2$; related to intensity

recall: *integrated* intensity is

$$I = \langle \vec{S} \cdot \vec{n} \rangle = \frac{c}{4\pi t} \int E^2(t) dt = \frac{c}{t} \int |E(\omega)|^2 dt \quad (48)$$

to *monochromatic intensity*

$$J_\omega = \frac{c |E(\omega)|^2}{t} \quad (49)$$

and since $\vec{E} = -1/c \partial_t \vec{A} = -i\omega/c \vec{A}$

$$J_\omega = \frac{\omega^2}{c t} |A(\omega)|^2 \quad (50)$$

and thus we see that $w_{fi} \propto |A(\omega)|^2$

implies $w_{fi} \propto J_\omega$, as expected for absorption!

also: what about w_{if} , for $f \rightarrow i$?

finally, for the transition probability per unit time for $i \rightarrow f$ we have

$$w_{fi} = \frac{4\pi^2 e^2}{m_e c^2} \frac{J(\omega_{fi})}{\omega_{fi}^2} \left| \langle f | e^{i\vec{k} \cdot \vec{r}} \mathbf{e} \cdot \sum_j \nabla_j | i \rangle \right|^2 \quad (51)$$

about the probability for $f \rightarrow i$?
 the same except now $\langle i | e^{i\vec{k} \cdot \vec{r}} \mathbf{e} \cdot \sum_j \nabla_j | i \rangle$
 but integrating by parts, can show

$$w_{if} = w_{fi} \quad (52)$$

principle of detailed balance

now: evaluate operator $e^{i\vec{k} \cdot \vec{r}} \mathbf{e} \cdot \sum_j \nabla_j$

the heart of the transition probability is
the matrix element $\int \psi_f^* e^{i\vec{k}\cdot\vec{r}} \mathbf{e} \cdot \sum_j \nabla_j \psi_i d^3x$

the wavenumber $k = \omega/c = \Delta E/\hbar c$

and the atomic wavefunctions are significant on scales $\sim a_0$

so: $\vec{k} \cdot \vec{r} \sim ka_0 \sim a_0 \Delta E/\hbar c \sim Zv/c \ll 1$

thus we write

$$e^{i\vec{k}\cdot\vec{r}} = 1 + i\vec{k} \cdot \vec{r} - \frac{1}{2}(\vec{k} \cdot \vec{r})^2 + \dots \quad (53)$$

and we approximate $e^{i\vec{k}\cdot\vec{r}} \approx 1$

Q: when would we be interested in the higher order terms?

we see that $e^{i\vec{k}\cdot\vec{r}} = 1 + i\vec{k}\cdot\vec{r} + \dots$

is an expansion in v/c

and we recall $v/c \ll 1$ for atoms with moderate $Z \ll 137$

lesson: expansion is dominated by first nonzero term

- $(kr)^0$ term: electric dipole approximation (more soon on this)
dominates unless identically zero, then
- $(kr)^1$ term: electric quadrupole approximation
and comparable magnetic dipole term ($B \sim v/c E$)
- $(kr)^2$ term: electric octupole, magnetic quadrupole

Note that to describe these terms,

have to modify Schrödinger equation to appropriate order in v/c

The Dipole Approximation

putting $e^{i\vec{k}\cdot\vec{r}} \approx 1$, the matrix element is

$$\int \psi_f^* \mathbf{e} \cdot \sum_j \nabla_j \psi_i d^3x = \frac{1}{i\hbar} \langle \mathbf{e} \cdot \hat{\mathbf{p}}_j \rangle_{fi} \quad (54)$$

i.e., related to the expected momentum of electron j

to bring this into a more familiar form, we note the basic quantum operator relationship

$$\hat{r}_j \hat{p}_j^2 - \hat{p}_j^2 \hat{r}_j = 2 i \hbar \hat{p}_j \quad (55)$$

and so given the atomic Hamiltonian

$$\hat{H}_0 = \frac{1}{2m_e} \sum_j \hat{\vec{p}}_j^2 + V(\hat{\vec{r}}_1, \hat{\vec{r}}_2, \dots, \hat{\vec{r}}_N) \quad (56)$$

we have

$$\hat{\vec{r}}_j \hat{H}_0 - \hat{H}_0 \hat{\vec{r}}_j = i \frac{\hbar \hat{\vec{p}}_j}{m_e} \quad (57)$$

a special case of the general result $-i\hbar\partial_t\hat{A} = [\hat{H}, \hat{A}]$

and so we have

$$\frac{1}{i\hbar} \langle \mathbf{e} \cdot \hat{\vec{p}}_j \rangle_{fi} = \frac{m_e}{\hbar^2} \int \psi_f^* \mathbf{e} \cdot (\hat{\vec{r}}_j H_0 - H_0 \hat{\vec{r}}_j) \psi_i d^3x \quad (58)$$

$$= \frac{m_e(E_i - E_f)}{\hbar^2} \int \psi_f^* \mathbf{e} \cdot \hat{\vec{r}} \psi_i d^3x \quad (59)$$

thus the transition rate is

$$w_{fi} = \frac{4\pi^2}{\hbar^2 c} \left| \langle \mathbf{e} \cdot \vec{d} \rangle_{fi} \right|^2 \mathcal{J}(\omega_{fi}) \quad (60)$$

where the **electric dipole operator** is

$$\vec{d} = e \sum_j r_j \quad (61)$$

note that generally we have atoms in random orientations so taking the angle average, we have

$$\langle |\mathbf{e} \cdot \vec{d}_{fi}|^2 \rangle = \frac{1}{3} |d_{fi}|^2 \quad (62)$$

where

$$|d_{fi}|^2 \equiv \vec{d}_{fi}^* \cdot \vec{d}_{fi} = |(d_x)_{fi}|^2 + |(d_y)_{fi}|^2 + |(d_z)_{fi}|^2 \quad (63)$$

Electric Dipole Transition Rate

the electric dipole transition rate is thus

$$\langle w_{fi} \rangle = \frac{4\pi^2}{3c\hbar^2} |d_{fi}|^2 \mathcal{J}(\omega_{fi}) \quad (64)$$

thus the Einstein absorption coefficient for $\ell \rightarrow u$ (“lower to upper”) is

$$\langle w_{lu} \rangle = B_{lu} J(\nu_{lu}) \quad (65)$$

where $J(\nu_{lu}) = \mathcal{J}(\nu_{lu})/4\pi$ since intensity is in one direction and $\mathcal{J}(\nu_{lu}) = \mathcal{J}(\omega_{lu}) d\omega/d\nu = 2\pi\mathcal{J}(\omega_{lu})$, so

$$\langle w_{lu} \rangle = \frac{1}{2} B_{lu} \mathcal{J}(\omega_{lu}) \quad (66)$$

and we can now find all three Einstein coefficients Q : *how?*

Einstein Coefficients

the Einstein coefficients in the electric dipole approximation are:

- true *absorption*

$$B_{lu} = \frac{8\pi^2}{3c\hbar^2} |d_{lu}|^2 = \frac{32\pi^4}{3ch} |d_{lu}|^2 \quad (67)$$

for *non-degenerate atomic levels* with $g_l = g_u = 1$ we have

- *stimulated emission*

$$B_{ul} = B_{lu} \quad (68)$$

- *spontaneous emission*

$$A_{ul} = \frac{2\nu^3}{c^2h} B_{lu} = \frac{64\pi^4 \nu_{ul}^3 |d_{ul}|^2}{3c^3h} \quad (69)$$

this gives (at least in principle) a direct means to connect

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the radiative coefficients j_ν and α_ν

to the atomic properties encoded in the dipole moment d_{ul}