Astronomy 501: Radiative Processes

Lecture 3

Aug 31, 2018

Announcements:

- **Problem Set 1** posted, due at **start of class** next Friday
- you may speak to me, the TA, and other students
  but you must *understand* your own answers
  and write them *yourself* and *in your own words*
  you may *not* consult old 501 problem sets/solutions
- Typo in printout: 3(b) should read "Evaluate $I_\lambda(s)$"

Last time:

- a blizzard of definitions!

Q: what is intensity? how does it differ from flux?

Q: what is intensity in ordinary experience/language?

Q: what is specific intensity? average intensity?

On Frequency and Wavelength

For most of the course, we will describe specific intensity using $I_\nu \equiv dI/d\nu$, i.e., in "frequency space"

But we could as well use $I_\lambda \equiv dI/d\lambda$: "wavelength space"

Of course, the two are related: in $(\nu, \nu + d\nu)$
the intensity $I_\nu \, d\nu$ is *equal to* $I_\lambda \, d\lambda$
where $(\lambda, \lambda + d\lambda)$ is the *corresponding* wavelength interval:
i.e., $\nu = c/\lambda$, and $d\nu = -c \, d\lambda/\lambda^2$

Thus the two intensity descriptions differ by a change of variable and thus by a Jacobian factor:

$$I_\lambda = \left| \frac{d\nu}{d\lambda} \right| \; I_\nu = \frac{c}{\lambda^2} I_{\nu(\lambda)} \tag{1}$$

- the same Jacobian factor is needed for F_λ, u_λ, etc.
- note that $\lambda I_\lambda = \nu I_\nu$: both give the intensity per unit log interval $|d\lambda/\lambda| = |d\nu/\nu|$; good to show on plots!

Photon Number

when using the photon picture of light
the basic units are *counts = number of photons*
where for monochromatic photons, $d\mathcal{E} = E_\nu \, dN = h\nu \, dN$

$\rightarrow$ useful to introduce the specific *number* intensity

$$\mathcal{I}_\nu = \frac{dN_\gamma}{dt \; dA \; d\Omega \; d\nu} = \frac{1}{h\nu} \; \frac{d\mathcal{E}}{dt \; dA \; d\Omega \; d\nu} = \frac{I_\nu}{h\nu} \tag{2}$$

and specific *number* flux

$$\Phi_\nu = \int \mathcal{I}_\nu \; \cos\theta \; d\Omega = \frac{1}{h\nu} \int I_\nu \; \cos\theta \; d\Omega = \frac{F_\nu}{h\nu} \tag{3}$$

Momentum Flux

consider the flux of photon *momentum*
in direction normal to area dA

For photons in solid angle $d\Omega$, from direction angle θ
contribution to *number flux* is $d\Phi_\nu = I_\nu / h\nu \ \cos\theta \ d\Omega$

photon momentum $p_\nu = h\nu/c$ has normal component
$p_{\nu,\perp} = h\nu/c \ \cos\theta$

photon momentum flux $\perp$ surface is **radiation pressure**

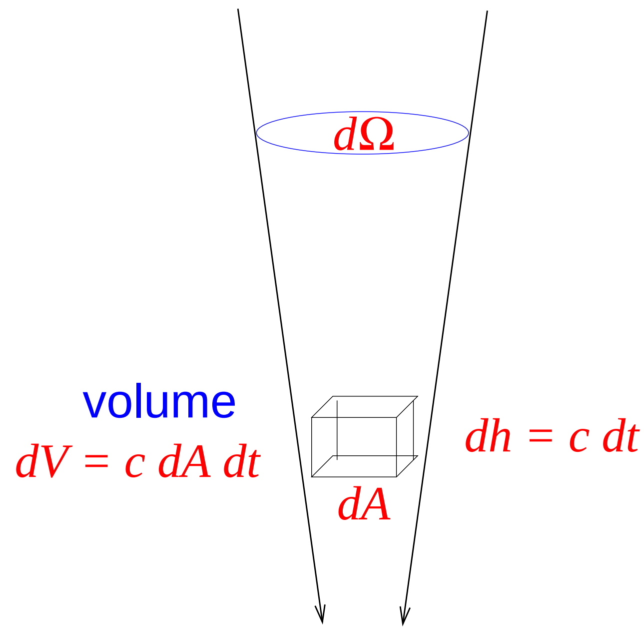
$$P_\nu = \int p_{\nu,\perp} \ d\Phi_\nu = \frac{1}{c} \int I_\nu \ \cos^2\theta \ d\Omega \tag{4}$$

for *isotropic* radiation

$$P_\nu^{\rm iso} = 2\pi \frac{I_\nu^{\rm iso}}{c} \int_{-1}^{1} \mu^2 d\mu = \frac{4\pi}{3} \ \frac{I_\nu^{\rm iso}}{c} \tag{5}$$

Energy Density

consider a bundle of rays passing through a small volume dV



energy density $u_\nu(\Omega)$ for bundle defined by $d\mathcal{E} = u_\nu(\Omega)\, d\Omega\, dV$

but $dV = dA\, dh$, and flux thru height dh in time $dt = dh/c$, so
$dV = c\, dA\, dt$

thus we have

$$d\mathcal{E} = c\ u_\nu(\Omega)\ dA\ dt\ d\Omega \qquad (6)$$

but by definition $d\mathcal{E} = I_\nu\ dA\ dt\ d\Omega$, so

$$u_\nu(\Omega) = \frac{I_\nu}{c} \qquad (7)$$

specific energy density in bundle in solid angle $d\Omega$

$$u_\nu(\Omega) = \frac{I_\nu}{c} \tag{8}$$

so total energy density is

$$\begin{aligned} u_\nu &= \int u_\nu \ d\Omega & (9) \\ &= \frac{1}{c}\int I_\nu \ d\Omega & (10) \\ &= \frac{4\pi J_\nu}{c} & (11) \end{aligned}$$

we can similarly find the photon specific **number density**

$$n_\nu = \frac{u_\nu}{h\nu} = \frac{4\pi J_\nu}{hc\nu} \tag{12}$$

Radiation Equation of State

recall: for isotropic radiation, pressure is momentum flux

$$P_\nu^{\rm iso} = \frac{4\pi}{3}\frac{I_\nu^{\rm iso}}{c} = \frac{u_\nu^{\rm iso}}{3} \tag{13}$$

pressure is 1/3 energy density, at each frequency!

note: relationship between pressure and (energy) density is an **equation of state**

thus people (=cosmologists) generalize this: $P = wu$
with w the "equation of state parameter"
we find: for isotropic radiation, $w_{\rm rad} = 1/3$

Integrated Intensity, Flux, Energy Density

specific intensity is per unit frequency: $I_\nu = dI/d\nu$
total or integrated intensity sums over all frequencies:

$$I = \int I_\nu \; d\nu \tag{14}$$

similarly, can define integrated flux

$$F = \int F_\nu \; d\nu \tag{15}$$

and integrate number and energy densities

$$n = \int n_\nu \; d\nu \tag{16}$$

$$u = \int u_\nu \; d\nu \tag{17}$$

∞ *Q: what if we use a broadband filter? Examples?*

Cosmic Color Wheel: Filtered Light

we measure using a broadband filter (“color”)
with a finite passband window

write *transmission fraction or probability* for photons at ν

$$W(\nu) = \left(\frac{\text{transmitted light}}{\text{incident light}} \right)_\nu \in [0, 1] \tag{18}$$

e.g., the classic $UBVGRIZ...$, or $ugrizY$

Q: who uses these? `www: transmission curves`

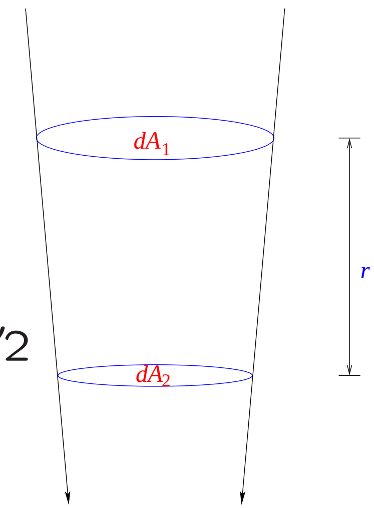
Then for band i, can define intensity

$$I_i = \int_{\text{band } i} I_\nu \; d\nu = \int W_i(\nu) \; I_\nu \; d\nu \tag{19}$$

and similarly for color flux $F_i = \int W_i F_\nu \; d\nu$ etc

Constancy of Specific Intensity in Free Space

in free space: no emission, absorption, scattering,
consider rays normal to areas dA_1 and dA_2 separated by a distance r



energy flow is conserved, so

$$d\mathcal{E}_1 = I_{\nu_1}\, dA_1\, dt\, d\Omega_1\, d\nu_1 = d\mathcal{E}_2 = I_{\nu_2}\, dA_2\, dt\, d\Omega_2\, d\nu_2$$

- as seen by dA_1, the solid angle $d\Omega_1$ subtended by dA_2 is $d\Omega_1 = dA_2/r^2$, and similarly $d\Omega_2 = dA_1/r^2$
- and in free space $d\nu_1 = d\nu_2$, so:

$$I_{\nu_1} = I_{\nu_2} \tag{20}$$

$$I_{\nu_1} = I_{\nu_2} \tag{21}$$

thus: in free space, the intensity is constant along a ray
that is: intensity of an object in free space
is *the same* anywhere along the ray

so along a ray in free space: $I_\nu =$ constant
or along small increment ds of the ray's path

$$\frac{dI_\nu}{ds} \stackrel{\text{free}}{=} 0 \tag{22}$$

this means: when viewing an object across free space,
the *intensity of the object is constant*
regardless of distance to the object!
$\Rightarrow$ **conservation of surface brightness**

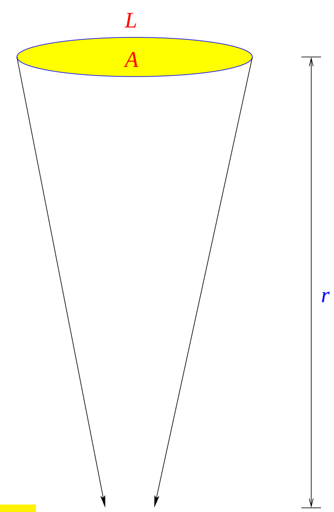
this is huge! and very useful!
Q: what is implied? how can this be true–what about inverse square law? everyday examples?

Conservation of Surface Brightness

consider object in free space at distance r
with luminosity L and project area $A \perp$ to sightline

flux from source follows usual inverse square

$$F = \frac{L}{4\pi r^2} \tag{23}$$



but *intensity* is flux *per solid angle*
and since $\Omega = A/r^2$, we have

$$I = \frac{F}{\Omega} = \frac{L/4\pi r^2}{A/r^2} = \frac{L}{4\pi A} \tag{24}$$

surface brightness is independent of distance!

and note $I = L/4\pi A$: intensity really is surface brightness
i.e., brightness per unit surface area and solid angle

Consequences of Surface Brightness Conservation

resolved objects in free space
have *same* I at all distances

- Sun's brightness at surface is same as you see in sky
  but at surface subtends 2π steradian – yikes!

- similar planetary nebulae or galaxies all have similar I
  regardless of distance

- people and objects across the room don't look $1/r^2$ dimmer
  than things next to you
  fun exercise: when in your everyday life
  do you actually experience the inverse square law for flux?

Adding Sources

matter can act as source and as sink for propagating light

the light energy added by glowing **source** in small volume dV, into a solid angle $d\Omega$, during time interval dt, and in frequency band $(\nu, \nu + d\nu)$, is written

$$d\mathcal{E}_{\rm emit} = j_\nu \; dV \; dt \; d\Omega \; d\nu \tag{25}$$

defines the **emission coefficient**

$$j_\nu = \frac{d\mathcal{E}_{\rm emit}}{dV \; dt \; d\Omega \; d\nu} \tag{26}$$

- power emitted per unit volume, frequency, and solid angle
- cgs units: $[j_\nu] = [\text{erg cm}^{-3} \text{ s}^{-1} \text{ sr}^{-1} \text{ Hz}^{-1}]$
- similarly can define j_λ, and integrated $j = \int j_\nu d\nu$

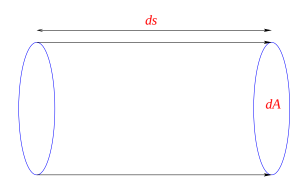
for *isotropic* emitters,
or for distribution of randomly oriented emitters, write

$$j_\nu = \frac{q_\nu}{4\pi} \tag{27}$$

where q_ν is radiated power per unit volume and frequency

sometimes also define *emissivity* $\epsilon_\nu = q_\nu / \rho$
energy emitted per unit freq and mass, with ρ =mass density

beam of area dA going distance ds
has volume $dV = dA \ ds$



so the *energy change* is $d\mathcal{E} = j_\nu \ ds \ dA \ dt \ d\Omega \ d\nu$
and the *intensity change* is

$$dI_\nu \overset{\text{sources}}{=} j_\nu \ ds \tag{28}$$

Adding Sinks

as light passes through matter, energy can also be lost due to scattering and/or absorption

we *model* this as follows:

$$dI_\nu = -\alpha_\nu \ I_\nu \ ds \tag{29}$$

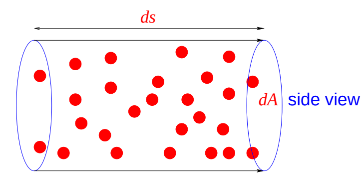
features/assumptions:

- losses proportional to distance ds traveled
  Q: why is this reasonable?
- losses proportional to intensity
  Q: why is this reasonable?
- defines energy loss per unit pathlength, i.e.,
  absorption coefficient α_ν
  Q: units/dimensions of α_ν?

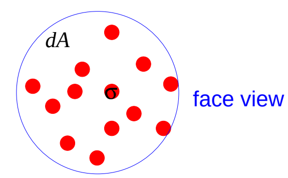
Absorption Cross Section

consider "absorbers" with a number density n_{a}
each of which presents the beam with an
effective *cross-sectional area* σ_ν

over length ds, number of absorbers is
$dN_{\text{a}} = n_{\text{a}}\, dA\, ds$



a "dartboard problem"– over beam area dA
total "bullseye" area: $\sigma_\nu dN_{\text{a}} = n_{\text{a}}\sigma_\nu\, dA\, ds$



so absorption *probability* is

$$dP_{\text{abs}} = \frac{\text{total bullseye area}}{\text{total beam area}} = n_{\text{a}}\ \sigma_\nu\ ds \qquad (30)$$

Q: for what length ds does $P_{\text{abs}} \rightarrow 1$?
Q: physical significance of $n_{\text{a}}\sigma_\nu$?

Cross Sections, Mean Free Path, and Absorption

absorption probability large when photon travels **mean free path**

$$\ell_{\rm mfp} = \frac{1}{n_{\rm a}\sigma_\nu} \tag{31}$$

so we can write $dP_{\rm abs} = ds/\ell_{\rm mfp}$

and thus beam energy change is

$$d\mathcal{E} = -dP_{\rm abs}\mathcal{E} = -n_{\rm a}\sigma_\nu I_\nu \; ds \; dA \; dt \; d\Omega \; d\nu \tag{32}$$

which must lead to an intensity change

$$dI_\nu \stackrel{\rm abs}{=} -n_{\rm a} \; \sigma_\nu \; I_\nu \; ds \tag{33}$$

Q: and so?

$$dI_\nu \stackrel{\text{abs}}{=} -n_\text{a}\ \sigma_\nu\ I_\nu\ ds \tag{34}$$

has the expected form, and we identify the absorption coeffcient

$$\alpha_\nu = n_\text{a}\ \sigma_\nu = \frac{1}{\ell_\text{mfp}} \tag{35}$$

note that absorption depends on

- *microphysics* via the cross section σ_ν
- *astrophysics* via density n_abs of scatterers

often, write $\alpha_\nu = \rho\kappa_\nu$,
defines **opacity** $\kappa_\nu = (n/\rho)\sigma_\nu \equiv \sigma_\nu/m$
with $m = \rho/n$ the mean mass per absorber

Q: so what determines σ_ν? e.g., for electrons?

Cross Sections

Note that the absorption **cross section** σ_ν is
and *effective* area presented by absorber

for "billiard balls" = neutral, opaque, macroscopic objects
this is the same as the geometric size
but generally, cross section is *unrelated to geometric size*
e.g., electrons are point particles (?) but still scatter light

- so *generalize* our ideas so that
  $dI_\nu = -n_{\rm a}\ \sigma_\nu\ ds$ *defines* the cross section
- determined by the details of light-matter interactions
- can be–and usually is!–frequency dependent
- differ according to physical process
  the study of which will be the bulk of this course!

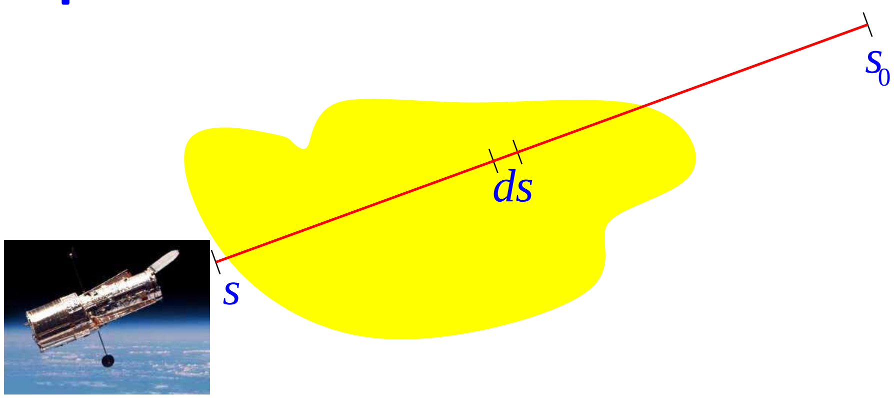
Note: "absorption" here is anything removing energy from beam
$\rightarrow$ can be true absorption, but also scattering

The Equation of Radiative Transfer

Now combine effects of sources and sinks
that change intensity as light propagates

$$\boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu} \tag{36}$$

equation of radiative transfer



- fundamental equation in this course
- *sources* parameterized via j_ν
- *sinks* parameterized via $\alpha_\nu = n_{\mathrm{a}}\,\sigma_\nu = \rho\kappa_\nu$

Transfer Equation: Limiting Cases

equation of radiative transfer:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \tag{37}$$

Sources but no Sinks

if sources exist (and are independent of I_ν) but no sinks: $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu \tag{38}$$

solve along path starting at sightline distance s_0:

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu \, ds' \tag{39}$$

- the *increment* in intensity is due to integral of sources *along sightline*
- for $j_\nu \to 0$: free space case and $I_\nu(s) = I_\nu(s_0)$: recover surface brightness conservation!

Special Case: Sinks but no Sources

if absorption only, no sources: $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \quad (40)$$

and so on a sightline from s_0 to s

$$I_\nu(s) = I_\nu(s_0)\ e^{-\int_{s_0}^{s} \alpha_\nu\ ds'} \quad (41)$$

- intensity *decrement* is *exponential*!
- exponent depends on line integral of absorption coefficient

useful to define **optical depth** via $d\tau_\nu \equiv \alpha_\nu\ ds$

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu\ ds' \quad (42)$$

and thus *for absorption only* $I_\nu(s) = I_\nu(s_0) e^{-\tau_\nu(s)}$