Astronomy 501: Radiative Processes

Lecture 6

Sept 10, 2018

Announcements:

- **Problem Set 2** due at start of class Friday

  3(e) hint: for c a constant, $\int x \sqrt{x^2 - c^2}\, dx = \frac{1}{3}(x^2 - c^2)^{3/2}$

Last time:

- optical thickness and observations

  Q: what do/don't we learn from optically thin image?

  one last example `www: SN1987A`
- radiation thermodynamics

  Q: blackbody integrated energy density $u(T)$?

Statistical Mechanics in a Nutshell

classically, **phase space** $(\vec{x}, \vec{p})$
completely describes particle state

Q: phase space lifestyle of single classical 1-D free body?
of single 1-D harmonic oscillator?
Q: a swarm of free bodies? oscillators?

but quantum mechanics $\rightarrow$ uncertainty $\Delta x \Delta p \geq \hbar/2$

semi-classically:
can show that a quantum particle must occupy
a *minimum* phase space "volume"
$(dx\ dp_x)(dy\ dp_y)(dz\ dp_z) = h^3 = (2\pi\hbar)^3$
per quantum state of fixed $\vec{p}$

Distribution Function

define "occupation number" or "distribution function" $f(\vec{x}, \vec{p})$:
number of particles in each phase space "cell"
Q: f range for fermions? bosons?
Q: what is f for one classical particle? many classical particles?

Given distribution function, total number of particles is

$$dN = g f(\vec{x}, \vec{p}) \frac{d^3\vec{x}\ d^3\vec{p}}{h^3} \quad (1)$$

where g is # internal states: spin/helicity, excitation
Q: $g(e^-)$? $g(\gamma)$? $g(p)$?

particle phase space occupation f determines bulk properties
Q: how? Hint–what's # particles per unit spatial volume?

Fermions: $0 \leq f \leq 1$ (Pauli)
Bosons: $f \geq 0$ $g(e^-) = 2s(e^-) + 1 = 2$ electron, same for p
$g(\gamma) = 2$ *(polarizations) photon*

Particle phase space occupation f determines bulk properties

Number density

$$n(\vec{x}) = \frac{d^3N}{d^3x} = \frac{g}{h^3} \int d^3\vec{p}\ f(\vec{p}, \vec{x}) \tag{2}$$

Q: this expressions is general–specialize to photons?

for photons $E = cp = h\nu$
so $d^3p = p^2\,dp\,d\Omega = h^3/c^3\ \nu^2\ d\nu\ d\Omega$

photon number density is thus

$$dn = \frac{2}{c^3}\nu^2\ f(\nu)\ d\nu\ d\Omega \tag{3}$$

and thus we have

$$\frac{dn_\nu}{d\Omega} = \frac{dn}{d\nu\ d\Omega} = \frac{2}{c^3}\nu^2\ f(\nu) \tag{4}$$

thus f *gives a general, fundamental description of photon fields*
the challenge is to find the physics that determines f
$\rightarrow$ spoiler alert: you have already seen a version of it!
but will see it again as the Boltzmann equation!

Note: distribution function $f(\nu)$ and specific intensity I_ν
are *equivalent* and *interchangeable descriptions*
Q: why? how do we get I_ν *from* $f(\nu)$*?*

Distribution Function and Observables

distribution function $f(\nu)$ is related to photon number via

$$\frac{dn_\nu}{d\Omega} = \frac{dN}{dV\ d\nu\ d\Omega} = \frac{2}{c^3}\nu^2\ f(\nu) \tag{5}$$

but we found that photon specific intensity is related to specific number density via

$$I_\nu = c\ h\nu\ \frac{dn_\nu}{d\Omega} \tag{6}$$

but this means that the two are related via

$$\boxed{I_\nu = \frac{2h}{c^2}\nu^3\ f(\nu)} \tag{7}$$

Equilibrium Occupation Numbers

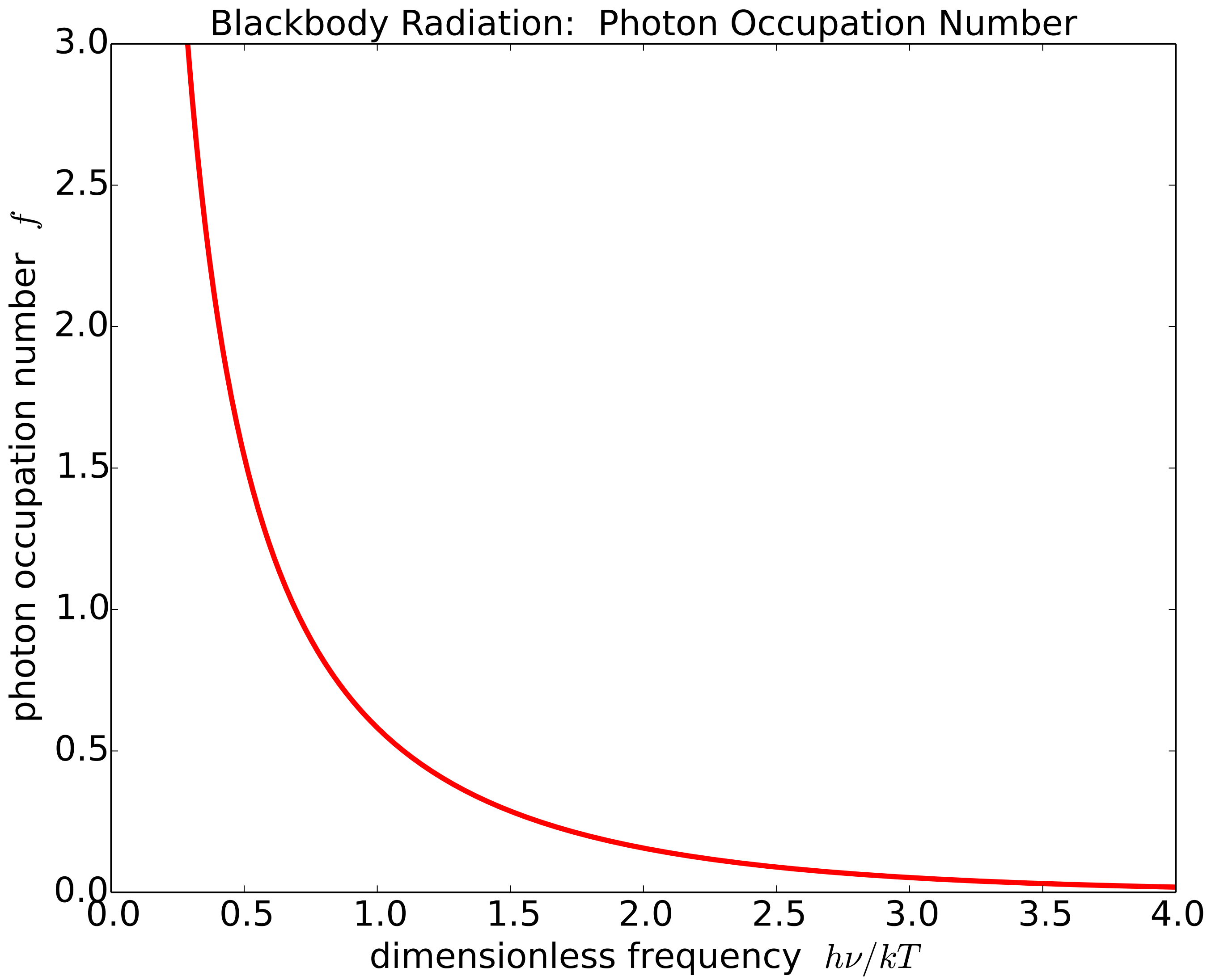
So far, totally general description of photon fields
no assumption of thermodynamic equilibrium

in thermodynamical equilibrium at T, the distribution function
is also the *occupation number*
i.e., average *number* of photons with freq ν

$$f(\nu, T) = \frac{1}{e^{h\nu/kT} - 1} \tag{8}$$

see derivation in today's Director's Cut Extras

Q: at fixed T, for which ν is f large? small?
Q: sketch of $f(\nu)$?
Q: what does this all mean physically?
Q: when is f zero?
Q: in which regime do we expect classical behavior? quantum?


Blackbody Radiation: Photon Occupation Number
photon occupation number f
dimensionless frequency $h\nu/kT$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

Blackbody Radiation Properties

Using the blackbody distribution function, we define

$$B_\nu(T) \equiv I_\nu(T) = \frac{2h}{c^2}\nu^3\ f(\nu, T) \tag{9}$$

and because $f(\nu, T) = 1/(e^{h\nu/kT} - 1)$, we have

$$B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{e^{h\nu/kT} - 1} \tag{10}$$

with h = Planck's constant, k = Boltzmann's constant

in wavelength space

$$B_\lambda(T) = 2hc^2\frac{\lambda^{-5}}{e^{hc/\lambda kT} - 1} \tag{11}$$

blackbody integrated intensity:

$$B(T) = \int B_\nu(T)\ d\nu = \int B_\lambda(T)\ d\lambda \tag{12}$$

$$= \frac{2\pi^4}{15} \frac{k^4 T^4}{c^3 h^3} = \frac{\sigma}{\pi} T^4 = \frac{c}{4\pi} a T^4 \tag{13}$$

blackbody flux

$$F_\nu(T) = \pi B_\nu(T) = \frac{2\pi h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{14}$$

$$F(T) = \pi B(T) \equiv \sigma T^4 = \frac{2\pi^5}{15} \frac{k^4 T^4}{c^2 h^3} \tag{15}$$

defines *Stefan-Boltzmann constant*

$$\sigma = \frac{2\pi^5}{15} \frac{k^4}{c^2 h^3} = 5.670 \times 10^{-5}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ K^{-4}} \tag{16}$$

integrated energy density

$$u_\nu(T) = \frac{4\pi B_\nu(T)}{c} = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1} \tag{17}$$

$$u(T) = \frac{4\pi B(T)}{c} = \frac{8\pi^5}{15}\frac{k^4T^4}{c^3h^3} \tag{18}$$

$$\equiv aT^4 = \frac{4\sigma}{c}T^4 \tag{19}$$

Stefan-Boltzmann radiation density constant

$$a = \frac{4\sigma}{c} = 7.56 \times 10^{-15}\ \text{erg cm}^{-3}\ \text{K}^{-4} \tag{20}$$

at last!

Q: to order of magnitude: integrated number density?

mean number density: dimensions $[n] = [\mathrm{length}^{-3}]$
can only depend on T, and physical constants h, c, k
can form only one length: $[hc/kT] = [\mathrm{length}]$
$\rightarrow$ expect $n \sim (hc/kT)^3$

photon number density

$$n_\nu(T) = \frac{4\pi B_\nu(T)}{hc\nu} = \frac{8\pi}{c^3}\frac{\nu^2}{e^{h\nu/kT} - 1} \quad (21)$$

$$n(T) = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3 \quad (22)$$

where $\zeta(3) = 1 + 1/2^3 + 1/3^3 + 1/4^3 + \cdots = 1.2020569\ldots$

Q: implications?

blackbody photon number density

$$n(T) = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3 \tag{23}$$

i.e., $n \propto T^3$

So if temperatures changes, photon number changes

blackbody photon number is not conserved

photons massless → can always make more!

if heat up, photon number increases

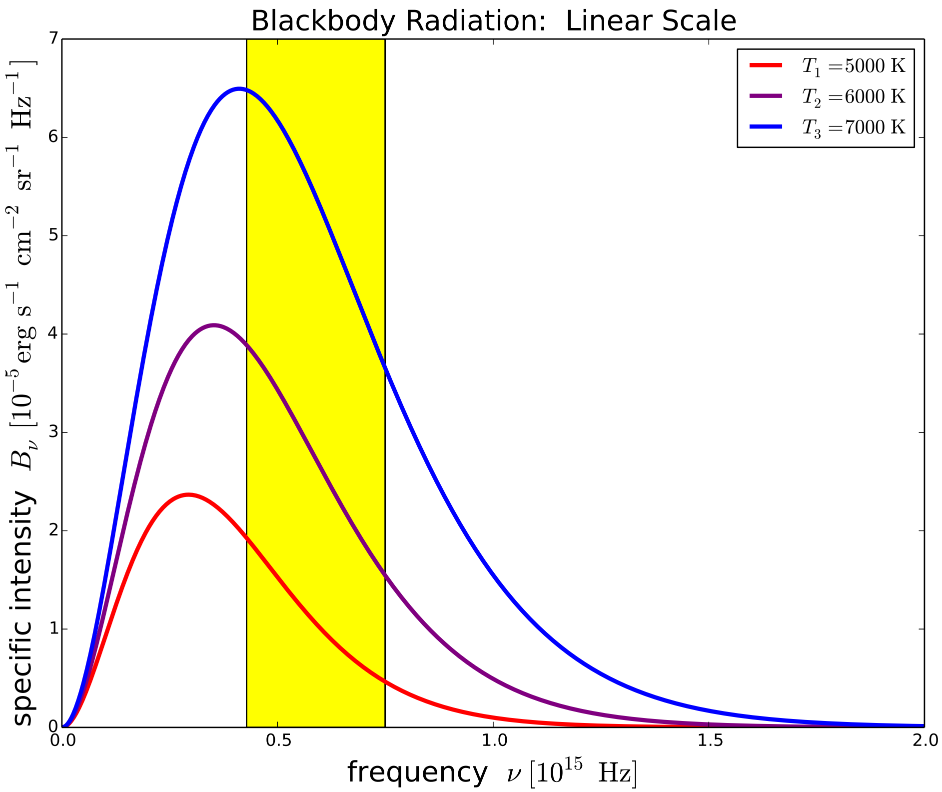
and spectrum relaxes to blackbody form

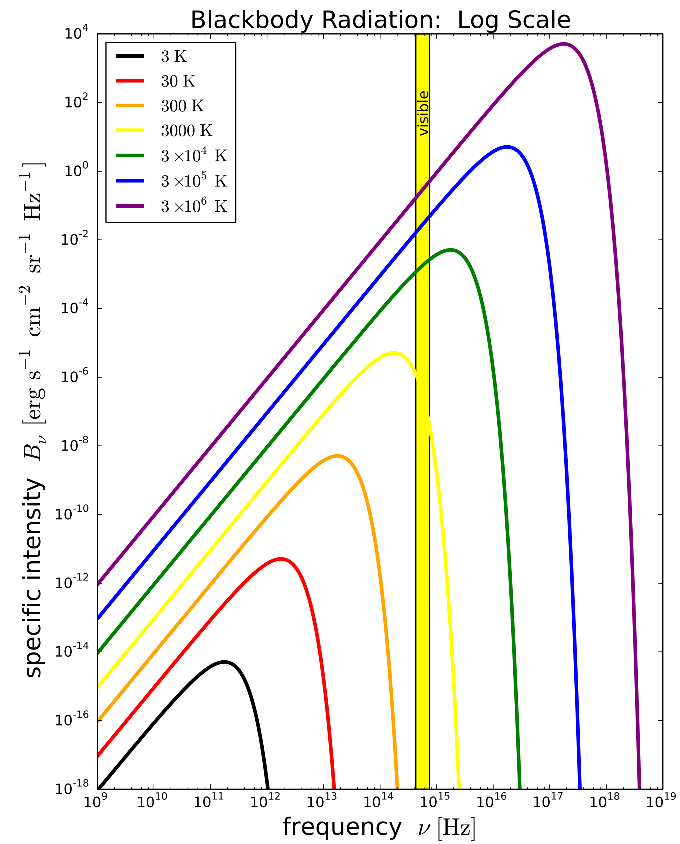
alternatively: given energy density $u \sim T^4$

and mean photon energy $\langle E \rangle \sim kT$

number density must be $n \sim T^3$

Blackbody Spectral Properties





plots of B_ν vs ν *Q: what strikes you?*

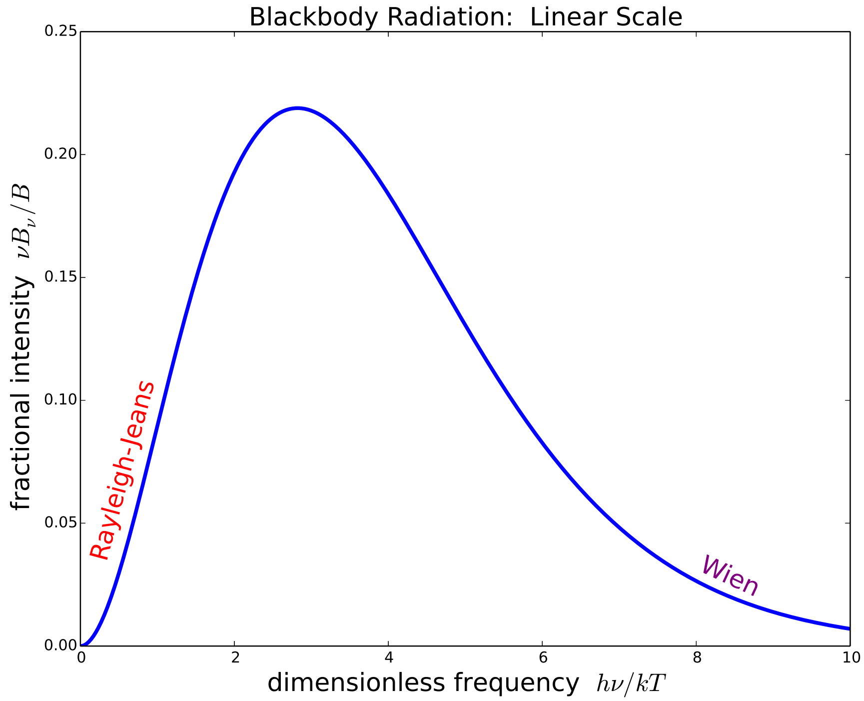
Blackbody Spectral Properties

at fixed ν, occupation number $\partial_T f(\nu, T) > 0$
$\rightarrow$ more photons, larger f for larger T
$\rightarrow$ more specific intensity, flux, energy density, at larger T
$\rightarrow$ slogan: *"blackbody spectra at different T never cross"*

natural energy scale kT, sets two limits

Rayleigh-Jeans limit $h\nu \ll kT$
occupation number $f(\nu) \rightarrow kT/h\nu \gg 1$
 many photons, expect classical behavior
specific intensity $I_\nu = 2h/c^2 \ \nu^3 f \rightarrow 2kT\,\nu^2/c^2$

- $I_\nu \propto \nu^2$: power-law scaling
- h does not appear in I_ν: classical behavior!



Wien limit $h\nu \gg kT$

occupation number $f(\nu) \to e^{-h\nu/kT} \ll 1$

photon starved: thermal bath cannot "pay energy cost"

specific intensity $I_\nu \to 2h\,\nu^3/c^2\ e^{-h\nu/kT}$

- exponentially damped due to quantum effects

Director's Cut Extras

Blackbody Photon Occupation Number

at a fixed temperature T and frequency ν
we want the distribution function f, i.e., the occupation number
i.e., the average number of photons with frequency ν

Boltzmann: probability of having state n of energy E_n
proportional to $p_n = e^{-E_n/kT}$

Planck: n photons have $E_n = n\,h\nu$, so $p_n = e^{-nx}$
with $x = h\nu/kT$

So average number is

$$f = \langle n \rangle = \frac{\sum_n n p_n}{\sum_n p_n} = \frac{\sum_n n e^{-nx}}{\sum_n e^{-nx}} \tag{24}$$

note that $\sum_n ne^{-nx} = -\partial_x \sum_n e^{-nx}$, so

$$f = -\partial_x \ln \left(\sum_n e^{-nx} \right) \tag{25}$$

but geometric series has sum

$$\sum_n e^{-nx} = \sum_n (e^{-x})^n = \frac{1}{1 - e^{-x}} \tag{26}$$

and thus

$$\begin{aligned} f &= -\partial_x \ln \frac{1}{1 - e^{-x}} = \partial_x \ln(1 - e^{-x}) \quad (27) \\ &= \frac{e^{-x}}{1 - e^{-x}} \quad (28) \end{aligned}$$

which gives

$$f(\nu, T) = \frac{1}{e^{h\nu/kT} - 1} \tag{29}$$

which was to be shewn

Average Energy per Blackbody Photon

only one way to form an energy
$\rightarrow$ expect $\langle E \rangle \sim kT$; exact result:

$$\langle E \rangle \equiv \frac{u(T)}{n(T)} \quad (30)$$

$$= \frac{\pi^4}{30\zeta(3)} kT = 2.701\, kT \quad (31)$$

c.f. nonrelativistic ideal gas: $\langle E \rangle_{\rm idealgas} = 3/2kT$

note: blackbody radiation has

$$\frac{P}{n\ kT} = \frac{\langle E \rangle}{3} = 0.900 \quad (32)$$

 c.f. nonrelativistic ideal gas: $P_{\rm idealgas}/n_{\rm idealgas}kT = 1$

Average Entropy per Blackbody Photon

mean entropy per photon:
entropy has units of Boltzmann's k
$\rightarrow$ expect $\langle S \rangle \sim k$; exact result

$$\langle S \rangle = \frac{s(T)}{n(T)} = \frac{4u(T)/3T}{n(T)} = \frac{4}{3}\frac{\langle E \rangle}{T} = 3.601\, k \tag{33}$$

temperature independent!

c.f. nonrelativistic ideal gas: entropy per particle
given by Sackur-Tetrode equation

$$\frac{s_{\text{idealgas}}}{n_{\text{idealgas}}} = k\left[\frac{5}{2} - \ln\left(\frac{n}{(2\pi mkT/h)^{3/2}}\right)\right] \tag{34}$$

nearly constant, but through logarithm term
weakly depends on T and density n