Astronomy 501: Radiative Processes

Lecture 7

Sept 12, 2018

Announcements:

- **Problem Set 2** due at start of class Friday
  3(e) hint: for c a constant, $\int x \sqrt{x^2 - c^2}\, dx = \frac{1}{3}(x^2 - c^2)^{3/2}$
- Physics Colloquim today: our own Prof. Jessie Shelton
  "The Higgs Portal onto the Dark Universe"

Last Time:

Q: blackbody B_ν spectrum form? characteristics?

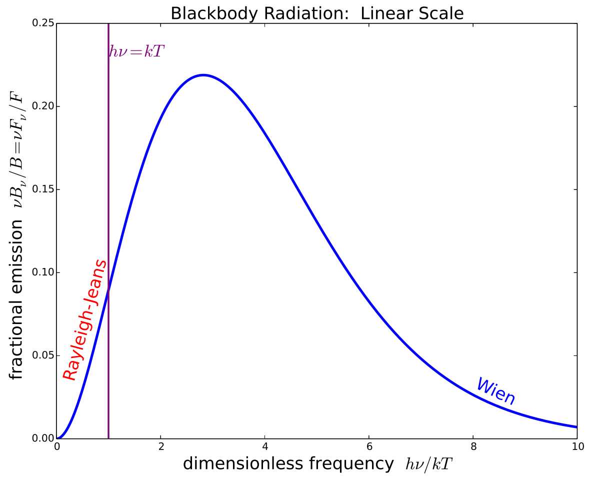
Q: limiting regimes?

Q: B_ν variation with T at fixed ν?

Q: when observing a resolved blackbody–what does B_ν give us?

Q: compare/contrast with unresolved blackbody?

Blackbody Radiation Highlights

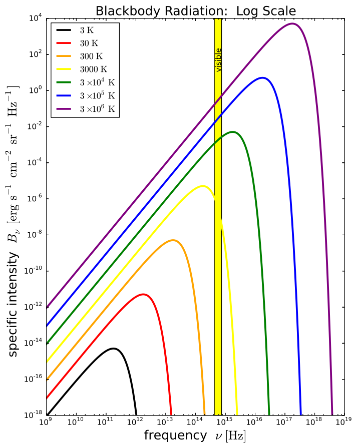


specific intensity: Planck function

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{1}$$

- low frequency $h\nu \ll kT$: **Raleigh-Jeans limit**

$$B_\nu \approx \frac{2h}{c^2} \nu^2 T$$



- high frequency $h\nu \gg kT$: **Wien limit**

$$B_\nu \approx \frac{2h}{c^2} \nu^3 \ e^{-h\nu/kT}$$

- at each ν: B_ν always increases with T
  blackbody intensity B_ν is *unique* for each T

Brightness Temperature

at each ν, blackbody $B_\nu(T)$ unique for each T
$\longrightarrow$ for resolved blackbody, measured B_ν gives T!

invert to define **brightness / antenna temperature**

$$B_\nu(T_{\rm b}) = I_\nu \tag{2}$$

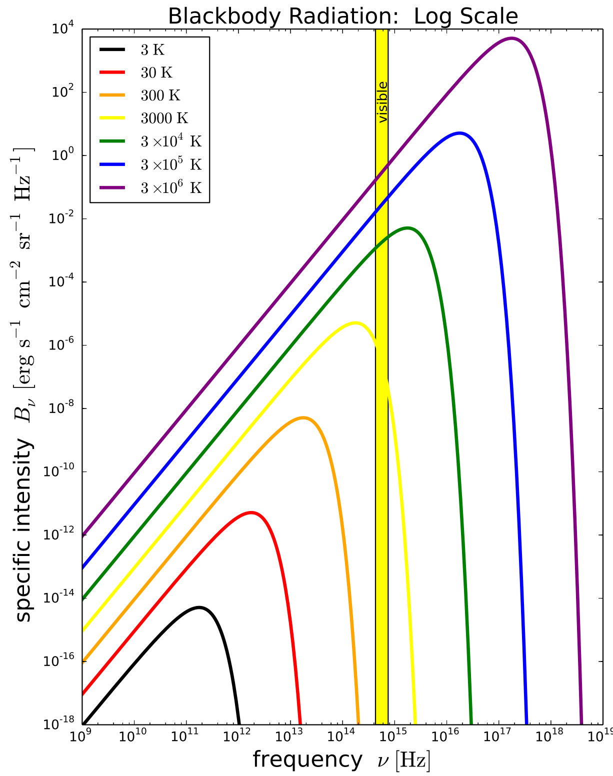
$$T_{\rm b}(\nu) = \frac{h\nu/k}{\ln\left(\frac{2h\nu^3}{c^2 I_\nu} + 1\right)} \tag{3}$$

$$\xrightarrow{h\nu \ll kT} \frac{c^2}{2k} \frac{I_\nu}{\nu^2} \tag{4}$$

note: *defined for all I_ν even if not blackbody!*
ω another way of characterizing intensity

brightness temperature

$$B_\nu(T_b) = I_\nu$$



Q: why is this useful?

Q: what does it mean for a nonthermal source?

`www: famous discovery`

`www: modern` T_b `results plotted`

Wien's Displacement Laws

for blackbodies,
specific intensity, and flux, and energy density have

$$I_\nu \propto F_\nu \propto u_\nu \propto \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{5}$$

at fixed T, these *spectra all peak at same frequency*

maximum when $x = h\nu/kT$ satisfies $x = 3(1 - e^{-x})$
$\rightarrow x_{\rm max} = 2.821439\ldots$, which gives

$$\frac{\nu_{\rm max}}{T} = x_{\rm max}\frac{kT}{h} = 5.88 \times 10^{10} \text{ Hz K}^{-1} \tag{6}$$

i.e., $\nu_{\rm max} \propto T$, as expected from dimensional analysis in wavelength space, $I_\lambda \propto \lambda^{-5}/(e^{hc/\lambda kT} - 1)$

maximum when $y = hc/\lambda kT$ satisfies $y = 5(1 - e^{-y})$
$\rightarrow y_{\rm max} = 4.9651\ldots$, which gives

$$\lambda_{\rm max}\, T = \frac{1}{y_{\rm max}} \frac{hc}{k} = 0.290 \text{ cm K} \tag{7}$$

i.e., $\lambda_{\rm max} \propto 1/T$, as expected from dimensional analysis

both versions of Wien's Law measure T: **color temperature**

crucial gotcha: **beware!** $\lambda_{\rm max} \neq c/\nu_{\rm max}$
Q: why?

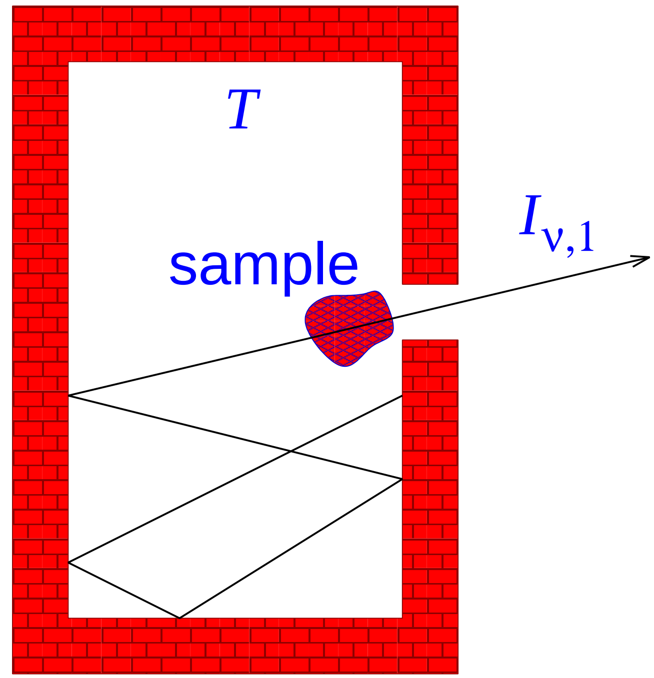
Thermal Radiation Transfer

Consider a cavity in thermodynamic equilibrium

place *small sample of matter* in cavity along peephole sightline

note: “sample,” most generally:

- *not* necessarily large
- *nor* optically thick!
- and might contain free ions, and/or bound states (atoms and molecules)



allow system to come into *thermodynamic equilibrium* at T

Q: effect on I_ν emerging from peephole?

Q: implications?

matter sample will emit and absorb radiation
according to its detailed composition,
and according to its state at T (i.e., ion, atom, molecule)
$\Rightarrow$ absorption $\alpha_\nu(T)$ and emission $j_\nu(T)$
both characteristic of the matter, *not universal!*

But in cavity+sample system, there is thermal equilibrium
$\rightarrow$ must still be blackbody emitter!
$\rightarrow$ *emitted radiation must have* $I_{\nu,\text{out}} = B_\nu(T)$

Yet radiation *incident* on sample in cavity
already was blackbody: $I_{\nu,\text{in}} = B_\nu(T)$

specific intensity change through sample

$$\frac{dI_\nu}{ds} = -\alpha_\nu (I_\nu - S_\nu) \tag{8}$$

Q: so what condition required for ray to maintain
$I_{\nu,\text{in}} = I_{\nu,\text{out}} = B_\nu(T)$ *through sample?*

Kirchhoff's Law for Thermal Emission

blackbody radiation $B_\nu(T)$ must remain unchanged
when traversing arbitrary matter with temperature T
$\Rightarrow$ demands that *all matter* at T *has source function*

$$S_\nu(T) = B_\nu(T) \tag{9}$$

Kirchhoff's Law for thermal emission

Physically: equilibrium forces *emission–absorption relation*:

$$j_\nu = \alpha_\nu \; B_\nu(T) \tag{10}$$

Consequences:

- a good (poor) emitter is a good (poor) absorber
- *thermal* emission has $S_\nu = B_\nu(T)$ and has

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu \tag{11}$$

Q: so when will thermal *source have* blackbody *spectrum?*

thermal source has $S_\nu(T) = B_\nu(T)$

as seen in the PS1 Olber's problem!

and so transfer equation becomes

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu$$

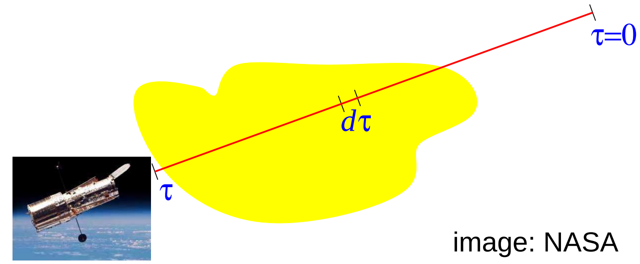


image: NASA

but since $B_\nu(T)$ is homogeneous, isotropic

if no background source, then $I_\nu(0) = 0$, and solution is

$$I_\nu(s) = \left(1 - e^{-\tau_\nu(s)}\right) B_\nu(T) \tag{12}$$

- if *optically thin*, then observe $I_\nu \approx \tau_\nu B_\nu(T) = j_\nu(T)\ \Delta s$
  intensity *not blackbody, and depends on emitter physics*
  `www: Cas A X-ray spectrum (Chandra)` *Q: what is Cas A?*

- if *optically thick*, then $I_\nu \to B_\nu$ as seen in PS1
  intensity relaxes to emitter-independent *blackbody* radiation

Build Your Toolbox: Imaging

Questions to ask about resolved astronomical images:

Q: what does it mean to be resolved?

Q: what questions to ask?

Q: how to answer?

Toolbox: Imaging

resolved images:

- *angular size > angular resolution*
- surface brigtness I_ν observable

Questions to ask

- What wavebands are in image?
- is the source optically thin or thick at these ν?

  optically thick: image shows surface

  $I_\nu \to S_\nu = j_\nu \ell_{\mathrm{mfp},\nu}$ at surface $\to B_\nu(T_{\mathrm{surf}})$ if thermal equilib

  optically thin: measure volume in projection

  $I_\nu \approx j_\nu \ \delta s$

Q: How to tell if optically thick/thin?

Thick or Thin?

How to tell if optically thick/thin? Non-trivial!

- use spectral info: compare image across ν
  does it follow blackbody?

- quantitatively: find brightness temperature $T_{\rm b}(\nu)$
  over multiple ν or colors
  if blackbody: uniform $T_{\rm b} = T$ across colors
  gives thermodynamic temperature
  otherwise: spectrum nonthermal,
  not all emission optically thick

Build Your Toolbox: Blackbody Radiation

emission physics: matter-radiation interactions

Q: physical conditions for blackbody radiation?

Q: physical nature of sources?

Q: spectrum characteristics?

Q: frequency range?

real/expected astrophysical sources of blackbody radiation

Q: what do we expect to emit blackbody radiation?

Q: relevant temperatures? EM bands?

Toolbox: Blackbody Radiation

emission physics

- physical conditions: *optically thick* objects *in thermodynamic equilibrium*
- physical sources: *solids, dense liquids or gasses*
- spectrum: continuum. nonzero at all frequencies
  single peak at $h\nu = 2.82kT$ (Wien's law)
  power law $I_\nu \sim \nu^2 T$ for $h\nu \ll kT$
  exponentical cutoff $I_\nu \sim \nu^3 e^{-h\nu/kT}$ $h\nu \gg kT$

astrophysical sources of blackbody radiation

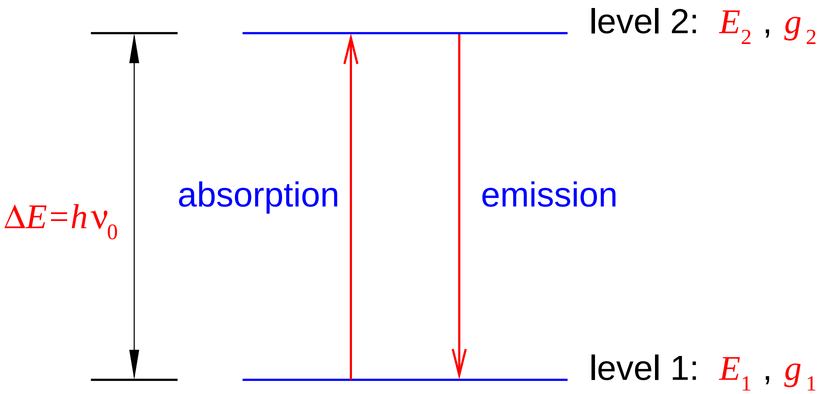
- *emitters: solids: dust, asteroids, planets, compact objects stars, some disks, the Universe*
- temperatures: *spans* 3K *to at least* 10^9 K
- EM bands: *microwave to MeV (and above?)*

Lines: Two-Level Systems

Two-Level Systems in Radiative Equilibrium

consider an ensemble of systems ("atoms") with

- **two discrete energy levels** E_1, E_2
- and degeneracies g_1, g_2
  i.e., a number g_1 of distinct states have energy E_1



in thermodynamic equilibrium at T, emission and absorption exchange energy with photon field

Q: when & why emit? absorb?
Q: connection between emission, absorption rates in ensemble?

Spontaneous Emission

in general, atoms in states with higher energy E_2
will *decay* to lower level E_1
photon of energy $\Delta E = E_2 - E_1 = h\nu_0$ will be emitted

transition can occur without influence of other atoms, photons
spontaneous emission: $X_2 \to X_1 + h\nu$

spontaneous emission rate per atom is
$E_2 \to E_1$ *transition rate per atom*:

$$\text{transition probability per unit time per atom} = A_{21} \qquad (13)$$

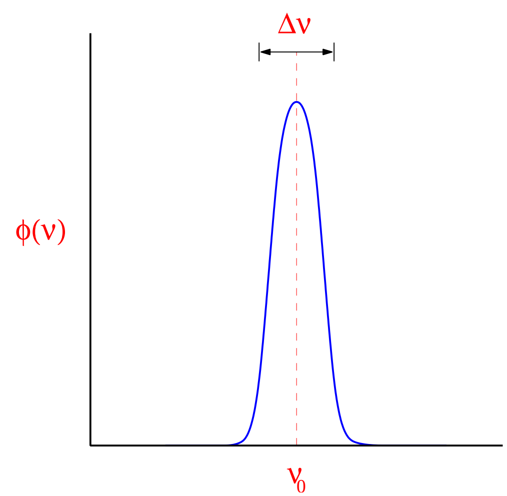
- "**Einstein** A" coefficient
- units $[A_{21}] = [\text{sec}^{-1}]$
- spontaneous: A_{21} independent of T
- but A_{21} *does depend on detailed atom properties*

Absorption

atoms in lower state E_1 only promoted to state E_2
by absorbing a photon of energy ΔE
$h\nu + X_1 \to X_2$

if levels were perfectly sharp, absorb only at $\Delta E = h\nu_0$
but in general, energy levels have *finite width*
i.e., *line energies "smeared out"* by some amount $h\,\Delta nu$
so transitions can be made by photons with frequencies
$\nu_0 - \Delta\nu \lesssim \nu \lesssim \nu_0 + \Delta\nu$

useful to define **line profile function** $\phi(\nu)$
with normalization $\int \phi(\nu)\ d\nu = 1$
e.g., Gaussian, Lorentzian, Voight functions



limiting case of sharp levels $\Delta\nu \to 0$:
$\phi(\nu) \to \delta(\nu - \nu_0)$

absorptions require ambient photons

thus absorption rate per atom *depends on photon field*
and ensemble *average absorption rate*
depends on average intensity

$$\bar{J} \equiv \int \phi(\nu)\ J_\nu\ d\nu \qquad (14)$$

limiting case of sharp levels: $\bar{J} \to J_{\nu_0}$

thus write average absorption rate as

$$\text{transition probability per time per atom} = B_{12}\bar{J} \qquad (15)$$

- "**Einstein B coefficient**"
- B_{12} is probability per time per intensity
- depends on atom and state details
- but *does not depend on T*

Stimulated Emission

Einstein postulated a new emission mechanism:
driven by photons with transition energy $\nu_0 - \Delta\nu \lesssim \nu \lesssim \nu_0 + \Delta\nu$
$h\nu + X_2 \rightarrow X_1 + 2h\nu$

I.e., the presence of transition photons creates "peer pressure"
"encourages" atoms in higher state to make transition
faster than they would spontaneously: **stimulated emission**
plausible? yes–photons interact with and perturb atoms

if stimulated emission exists, should also depend on $\bar{J}$
$\rightarrow$ rate per atom is

$$\text{transition probability per time per atom} = B_{21}\bar{J} \quad (16)$$

- note stimulated emission coefficient B_{21}
  can be *different from* absorption coefficient B_{12}
- if stimulated emission doesn't happen, would find $B_{21} = 0$

The Equilibrium Condition

In thermodynamic equilibrium, the numbers n_1, n_2 of atoms in each state do not change with time
$\longrightarrow$ total emission rate is equal to absorption rate

$$n_2 A_{21} + n_2 B_{21} \bar{J} = n_1 B_{12} \bar{J} \tag{17}$$

solve for ambient radiation field

$$\bar{J} = \frac{A_{21}/B_{21}}{n_1/n_2 \; B_{12}/B_{21} - 1} \tag{18}$$

in thermodynamic equilibrium, atom state populations follow *Boltzmann distribution*

$$\frac{n_1}{n_2} = \frac{g_1 e^{-E_1/kT}}{g_2 e^{-E_2/kT}} = \frac{g_1}{g_1} e^{(E_2 - E_1)/kT} = \frac{g_1}{g_1} e^{h\nu_0/kT} \tag{19}$$

and so

$$\bar{J} = \frac{A_{21}/B_{21}}{g_1/g_2 \; B_{12}/B_{21} \; e^{h\nu_0/kT} - 1} \tag{20}$$

thus we find that in equilibrium,
the mean intensity near ν_0 is

$$\bar{J} = \frac{A_{21}/B_{21}}{g_1/g_2 \; B_{12}/B_{21} \; e^{h\nu_0/kT} - 1} \tag{21}$$

but in equilibrium, and with narrow linewidth,
the mean intensity should be blackbody result:

$$\bar{J} \rightarrow B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} \tag{22}$$

Q: what condition(s) must hold to satisfy
both *equations for* any T*?*

because A and both B do not depend on T

the only way to have, at any T,

$$\bar{J} = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} = \frac{A_{21}/B_{12}}{g_1/g_2 \; B_{12}/B_{21} \; e^{h\nu/kT} - 1} \tag{23}$$

is to require two *Einstein relations*

$$A_{21} = \frac{2h\ \nu^3}{c^2} B_{21} \tag{24}$$

$$g_2 B_{21} = g_1 B_{12} \tag{25}$$

- these relations are independent of T:
  hold even without thermal equilibrium!
- $B_{21} \neq 0$: spontaneous emission exists!
  and typically has probability comparable to absorption!
  give it up for Big Al!

Two-Level Systems: Thermal Radiation

Now consider the two-level atom as a radiating system
What are the emission and absorption coefficients?

Emission Coefficient

spontaneous emission rate per atom in state 2: A_{21}
$\rightarrow$ rate per volume: $n_2 A_{21}$
$\rightarrow$ total power emitted per volume: $h\nu_0 \, n_2 A_{21}$
emission isotropic $\rightarrow$ power per volume per solid angle:
$h\nu_0 \, n_2 A_{21}/4\pi$ *Q: why?*

but still need *frequency spectrum*
of emitted radiation, i.e., *emission profile*

Q: simplest assumption?

simplest assumption (generally accurate):
$\rightarrow$ emission spectrum profile = absorption profile $\phi(\nu)$

and thus energy released in spontaneous emission is

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \ dV \ dt \ d\nu \ d\Omega \tag{26}$$

and thus the emission coefficient is

$$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \tag{27}$$

absorption coefficient

absorption rate per atom in level 1: $B_{12}\bar{J}$

thus energy absorbed is

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_1 B_{12} \bar{J}\ dV\ dt \tag{28}$$

but $4\pi\bar{J} = \int d\Omega \int I_\nu \phi(\nu)\ d\nu$, so

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_1 B_{12} I_\nu \phi(\nu)\ dV\ dt\ d\Omega\ d\nu \tag{29}$$

recall: path element ds in area dA has volume $dV = ds\ dA$ and so we find absorption coefficient

$$\alpha_{\mathrm{abs},\nu} = \frac{h\nu_0}{4\pi} n_1 B_{12} \phi(\nu) \tag{30}$$

...but we are not done! *Q: because...?*

stimulated emission

tempting to include this as additional emission term

but wait! stimulated emission depends on (average) intensity
$\longrightarrow$ formally more similar to absorption

formally better to treat stimulated emission as
a *negative absorption* term:

$$\alpha_{\mathrm{stim},\nu} = -\frac{h\nu_0}{4\pi} n_2 B_{21} \phi(\nu) \tag{31}$$

and then *(net) absorption coefficient*

$$\begin{aligned} \alpha_\nu &= \alpha_{\mathrm{abs},\nu} + \alpha_{\mathrm{stim},\nu} && (32) \\ &= \frac{h\nu_0}{4\pi}\phi(\nu)\ \left(n_1 B_{12} - n_2 B_{21}\right) && (33) \end{aligned}$$

Two-Level Radiation Transfer

Transfer equation for two-level atom

$$\frac{dI_\nu}{ds} = -\frac{h\nu_0}{4\pi}\phi(\nu)\ \left(n_1 B_{12} - n_2 B_{21}\right) I_\nu + \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \tag{34}$$

source function

$$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} \tag{35}$$

Einstein relations give

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12} \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) \tag{36}$$

$$S_\nu = \frac{2h\nu^3/c^2}{(n_1/n_2)(g_2/g_1) - 1} \tag{37}$$

a generalization of Kirchhoff's laws

these *do not assume thermal equilibrium!*

Q: interesting cases?

Local Thermodynamic Equilibrium

if atom levels are in thermodynamic equilibrium
then we have $n_1/n_2 = (g_1/g_2)e^{h\nu/kT}$ and

$$S_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} = B_\nu(T) \tag{38}$$

we recover the usual Kirchhoff's law! as we must!

and absorption term becomes

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12}\left(1 - e^{-h\nu/kT}\right) \tag{39}$$

i.e., "uncorrected" term minus stimulated emission correction

What if not in thermodynamic equilibrium?
then $n_1/n_1 \neq$ Boltzmann expression
emission is nothermal

Inverted Populations

two-level absorption coefficient is:

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12} \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) \tag{40}$$

note that the algebraic *sign*
  depends on population levels, i.e., on n_2/n_1
normally, lower level more populated: $n_1 > n_2$

If we can arrange or stumble upon a system where

$$\frac{n_1}{g_1} < \frac{n_2}{g_2} \tag{41}$$

i.e., an *inverted population*, then $\alpha_\nu < 0$!

Q: and then what happens to propagating light?
Q: examples?

Q: how might we arrange an inverted population?

Masers

if $\alpha_\nu < 0$, then propagating light has exponential *increase* in intensity!

stimulated emission causes a "cascade" of photons

in lab: create inverted populations of atoms
use mirrors to "recycle" stimulating photons
$\longrightarrow$ this is a laser! light amplification by stimulated emission of radiation

in cosmos: inverted populations of molecules
maser: microwave amplification by stimulated emission of radiation

how to create inversion?
need *nonthermal* mechanism to "pump" upper level