

Astronomy 501: Radiative Processes

Lecture 8

Sept 14, 2018

Announcements:

- **Problem Set 2** due by ulcred5pm today
- **Problem Set 3** due next Friday at start of class

Last time: Kirchhoff's law

Q: what is it?

Q: why might it seem like a miracle?

*Q: why might it **not** seem like a miracle?*

hint—consider emission/absorption if Kirchhoff not true?

*Q: what's the difference between a **thermal** radiation
and **blackbody** radiation?*

Kirchhoff's Law Recap

Kirchhoff: *any* matter in thermal equilibrium at T :

$$S_\nu(T) = B_\nu(T) \quad (1)$$

$$j_\nu(T) = \alpha_\nu(T) B_\nu(T) \quad (2)$$

- emission rate related to absorption rate
- good emitters are good absorbers

How does emission “know” to be related to absorption?

If emission exceeded absorption, matter loses energy
would cool until emission = absorption
condition of equilibrium enforces “detailed balance”

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thermal radiation: emitted by *any matter at T*

blackbody radiation: emitted *optically thick* matter at T

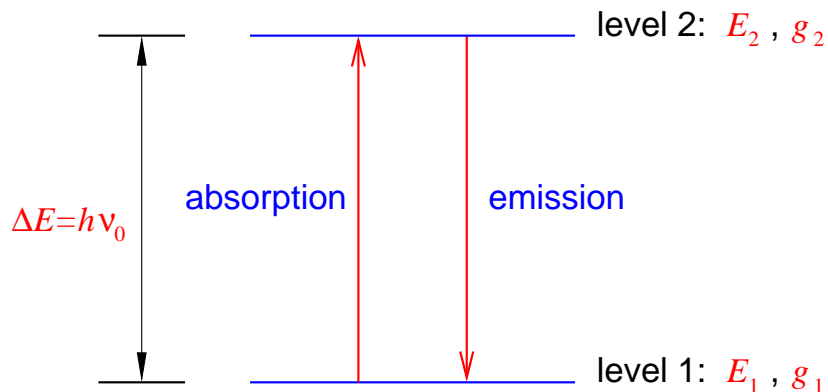
Two-Level Systems in Radiative Equilibrium

consider an ensemble of systems (“atoms”) with

- **two discrete energy levels** E_1, E_2

- and degeneracies g_1, g_2

i.e., a number g_1 of distinct states have energy E_1



in thermodynamic equilibrium at T , emission and absorption exchange energy with photon field

ω

Q: when & why emit? absorb?

Q: connection between emission, absorption rates in ensemble?

Spontaneous Emission

in general, atoms in states with higher energy E_2 will *decay* to lower level E_1
photon of energy $\Delta E = E_2 - E_1 = h\nu_0$ will be emitted

transition can occur without influence of other atoms, photons
spontaneous emission: $X_2 \rightarrow X_1 + h\nu$

spontaneous emission rate per atom is
 $E_2 \rightarrow E_1$ *transition rate per atom:*

$$\text{transition probability per unit time per atom} = A_{21} \quad (3)$$

- “**Einstein A**” coefficient
- units $[A_{21}] = [\text{sec}^{-1}]$
- spontaneous: A_{21} independent of T
- but A_{21} *does depend on detailed atom properties*

Absorption

atoms in lower state E_1 only promoted to state E_2 by absorbing a photon of energy ΔE



if levels were perfectly sharp, absorb only at $\Delta E = h\nu_0$ but in general, energy levels have *finite width*

i.e., *line energies "smeared out"* by some amount $h \Delta\nu$ so transitions can be made by photons with frequencies

$$\nu_0 - \Delta\nu \lesssim \nu \lesssim \nu_0 + \Delta\nu$$

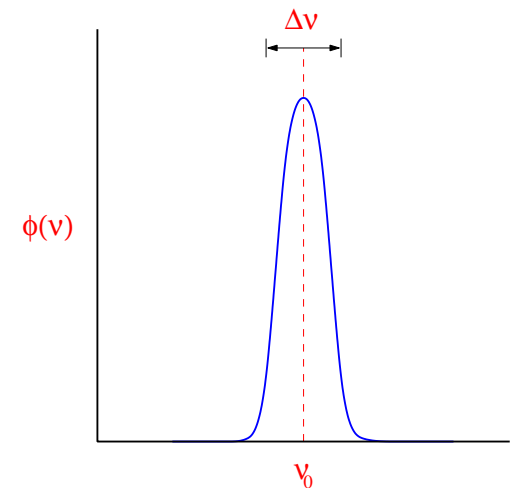
useful to define **line profile function** $\phi(\nu)$

with normalization $\int \phi(\nu) d\nu = 1$

e.g., Gaussian, Lorentzian, Voigt functions

⌚ limiting case of sharp levels $\Delta\nu \rightarrow 0$:

$$\phi(\nu) \rightarrow \delta(\nu - \nu_0)$$



absorptions require ambient photons

thus absorption rate per atom *depends on photon field*
and ensemble *average absorption rate*
depends on average intensity

$$\bar{J} \equiv \int \phi(\nu) J_\nu d\nu \quad (4)$$

limiting case of sharp levels: $\bar{J} \rightarrow J_{\nu_0}$

thus write average absorption rate as

$$\text{transition probability per time per atom} = B_{12} \bar{J} \quad (5)$$

- “**Einstein B coefficient**”
- B_{12} is probability per time per intensity
- ● depends on atom and state details
- but *does not depend on T*

Stimulated Emission

Einstein postulated a new emission mechanism:

driven by photons with transition energy $\nu_0 - \Delta\nu \lesssim \nu \lesssim \nu_0 + \Delta\nu$
 $h\nu + X_2 \rightarrow X_1 + 2h\nu$

I.e., the presence of transition photons creates “peer pressure” “encourages” atoms in higher state to make transition faster than they would spontaneously: **stimulated emission**
plausible? yes—photons interact with and perturb atoms

if stimulated emission exists, should also depend on \bar{J}
→ rate per atom is

$$\text{transition probability per time per atom} = B_{21}\bar{J} \quad (6)$$

- note stimulated emission coefficient B_{21}
can be *different from* absorption coefficient B_{12}
- if stimulated emission doesn't happen, would find $B_{21} = 0$

The Equilibrium Condition

In thermodynamic equilibrium, the numbers n_1, n_2 of atoms in each state do not change with time
→ total emission rate is equal to absorption rate

$$n_2 A_{21} + n_2 B_{21} \bar{J} = n_1 B_{12} \bar{J} \quad (7)$$

solve for ambient radiation field

$$\bar{J} = \frac{A_{21}/B_{21}}{n_1/n_2 \cdot B_{12}/B_{21} - 1} \quad (8)$$

in thermodynamic equilibrium, atom state populations follow *Boltzmann distribution*

$$\frac{n_1}{n_2} = \frac{g_1 e^{-E_1/kT}}{g_2 e^{-E_2/kT}} = \frac{g_1}{g_2} e^{(E_2 - E_1)/kT} = \frac{g_1}{g_2} e^{h\nu_0/kT} \quad (9)$$

and so

$$\bar{J} = \frac{A_{21}/B_{21}}{g_1/g_2 \cdot B_{12}/B_{21} \cdot e^{h\nu_0/kT} - 1} \quad (10)$$

thus we find that in equilibrium,
the mean intensity near ν_0 is

$$\bar{J} = \frac{A_{21}/B_{21}}{g_1/g_2 \frac{B_{12}}{B_{21}} e^{h\nu_0/kT} - 1} \quad (11)$$

but in equilibrium, and with narrow linewidth,
the mean intensity should be blackbody result:

$$\bar{J} \rightarrow B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} \quad (12)$$

Q: *what condition(s) must hold to satisfy
both equations for any T ?*

because A and both B do not depend on T
the only way to have, at any T ,

$$\bar{J} = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} = \frac{A_{21}/B_{12}}{g_1/g_2 \cdot B_{12}/B_{21} \cdot e^{h\nu/kT} - 1} \quad (13)$$

is to require two *Einstein relations*

$$A_{21} = \frac{2h \nu^3}{c^2} B_{21} \quad (14)$$

$$g_2 B_{21} = g_1 B_{12} \quad (15)$$

- these relations are independent of T :
hold even without thermal equilibrium!
- $B_{21} \neq 0$: spontaneous emission exists!
and typically has probability comparable to absorption!
give it up for Big AI!

Two-Level Systems: Thermal Radiation

Now consider the two-level atom as a radiating system
What are the emission and absorption coefficients?

Emission Coefficient

spontaneous emission rate per atom in state 2: A_{21}

→ rate per volume: $n_2 A_{21}$

→ total power emitted per volume: $h\nu_0 n_2 A_{21}$

emission isotropic → power per volume per solid angle:

$$h\nu_0 n_2 A_{21} / 4\pi \quad \text{Q: why?}$$

but still need *frequency spectrum*

of emitted radiation, i.e., *emission profile*

Q: *simplest assumption?*

simplest assumption (generally accurate):

→ emission spectrum profile = absorption profile $\phi(\nu)$

and thus energy released in spontaneous emission is

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) dV dt d\nu d\Omega \quad (16)$$

and thus the emission coefficient is

$$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \quad (17)$$

absorption coefficient

absorption rate per atom in level 1: $B_{12}\bar{J}$

thus energy absorbed is

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_1 B_{12} \bar{J} dV dt \quad (18)$$

but $4\pi\bar{J} = \int d\Omega \int I_\nu \phi(\nu) d\nu$, so

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_1 B_{12} I_\nu \phi(\nu) dV dt d\Omega d\nu \quad (19)$$

recall: path element ds in area dA has volume $dV = ds dA$ and so we find absorption coefficient

$$\alpha_{\text{abs},\nu} = \frac{h\nu_0}{4\pi} n_1 B_{12} \phi(\nu) \quad (20)$$

...but we are not done! Q: *because...?*

stimulated emission

tempting to include this as additional emission term

but wait! stimulated emission depends on (average) intensity

→ formally more similar to absorption

formally better to treat stimulated emission as a *negative absorption* term:

$$\alpha_{\text{stim},\nu} = -\frac{h\nu_0}{4\pi}n_2B_{21}\phi(\nu) \quad (21)$$

and then *(net) absorption coefficient*

$$\alpha_\nu = \alpha_{\text{abs},\nu} + \alpha_{\text{stim},\nu} \quad (22)$$

$$= \frac{h\nu_0}{4\pi}\phi(\nu) (n_1B_{12} - n_2B_{21}) \quad (23)$$

Two-Level Radiation Transfer

Transfer equation for two-level atom

$$\frac{dI_\nu}{ds} = -\frac{h\nu_0}{4\pi}\phi(\nu) (n_1B_{12} - n_2B_{21}) I_\nu + \frac{h\nu_0}{4\pi}n_2A_{21}\phi(\nu) \quad (24)$$

source function

$$S_\nu = \frac{n_2A_{21}}{n_1B_{12} - n_2B_{21}} \quad (25)$$

Einstein relations give

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu) n_1B_{12} \left(1 - \frac{n_2g_1}{n_1g_2} \right) \quad (26)$$

$$S_\nu = \frac{2h\nu^3/c^2}{(n_1/n_2)(g_2/g_1) - 1} \quad (27)$$

a generalization of Kirchhoff's laws

these *do not assume thermal equilibrium!*

Q: *interesting cases?*

Local Thermodynamic Equilibrium

if atom levels are in thermodynamic equilibrium
then we have $n_1/n_2 = (g_1/g_2)e^{h\nu/kT}$ and

$$S_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} = B_\nu(T) \quad (28)$$

we recover the usual Kirchhoff's law! as we must!

and absorption term becomes

$$\alpha_\nu = \frac{h\nu_0}{4\pi} \phi(\nu) n_1 B_{12} (1 - e^{-h\nu/kT}) \quad (29)$$

i.e., “uncorrected” term minus stimulated emission correction

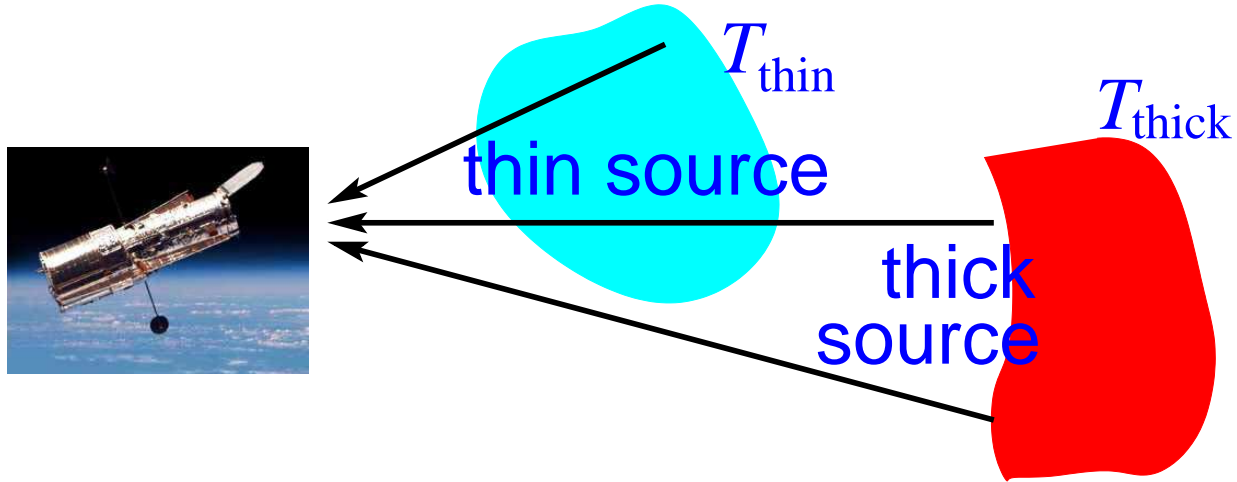
What if not in thermodynamic equilibrium?

then $n_1/n_2 \neq$ Boltzmann expression
emission is nonthermal

Kirchoff's Laws in Action

Consider matter including 2-level system

$\Delta E = h\nu_0$, in equilibrium at T



Q: What spectrum if optically thin (and not backlit)?

Q: What spectrum if optically thick (and no foreground matter)?

Q: astro examples?

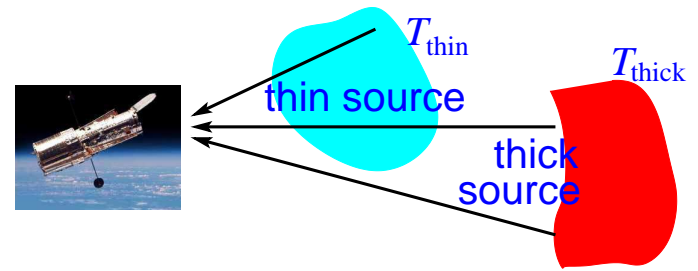
Kirchoff: Single Source

Kirchoff: thermal source has $j_\nu(T) = \alpha_\nu(T) B_\nu(T)$

2-level: $j_\nu \propto \alpha_\nu \propto \tau_\nu \propto \phi(\nu)$ line profile

single source at T

- optically thin: $I_\nu = j_\nu(T) \delta s$
see *line at ν_0 in emission*
- optically thick: $I_\nu = S_\nu = B_\nu$ *blackbody, Planck function*



Now consider same matter, but

- optically thick source at T_{thick}
- enshrouded by optically thin source at T_{thin}

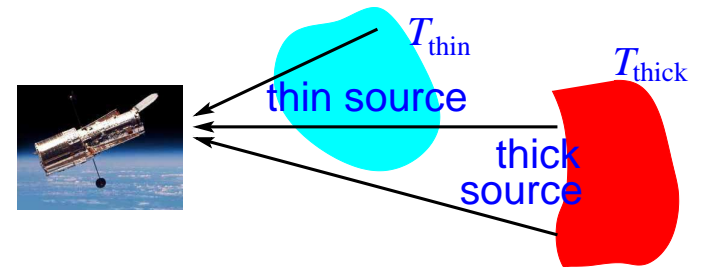
Q: *spectrum seen only through thin source?*

Q: *spectrum thru thick source if $T_{\text{thick}} > T_{\text{thin}}$? if $T_{\text{thick}} > T_{\text{thin}}$?*

Q: *astro examples of each case?*

two sources at T_{thick} and T_{thin}

- thin source only: emission line spectrum
- for thin source backlit by thick:



$$I_{\nu} \approx (1 - \tau_{\nu}) B_{\nu}(T_{\text{thick}}) + \tau_{\nu} B_{\nu}(T_{\text{thin}}) \quad (30)$$

$$= B_{\nu}(T_{\text{thick}}) + \tau_{\nu} [B_{\nu}(T_{\text{thin}}) - B_{\nu}(T_{\text{thick}})] \quad (31)$$

- $\nu \neq \nu_0$: $I_{\nu} \approx B_{\nu}(T_{\text{thick}})$
observe *Planck continuum* of thick source
- around ν_0 : $T_{\text{thick}} > T_{\text{thin}}$ gives *absorption line* in continuum
 $T_{\text{thick}} < T_{\text{thin}}$ gives *emission line* above continuum

Inverted Populations

two-level absorption coefficient is:

$$\alpha_\nu = \frac{h\nu_0}{4\pi} \phi(\nu) n_1 B_{12} \left(1 - \frac{n_2 g_1}{n_1 g_2} \right) \quad (32)$$

note that the algebraic *sign*

depends on population levels, i.e., on n_2/n_1
normally, lower level more populated: $n_1 > n_2$

If we can arrange or stumble upon a system where

$$\frac{n_1}{g_1} < \frac{n_2}{g_2} \quad (33)$$

i.e., an *inverted population*, then $\alpha_\nu < 0$!

Q: and then what happens to propagating light?

Q: examples?

Q: how might we arrange an inverted population?

Masers

if $\alpha_\nu < 0$, then propagating light has exponential *increase* in intensity!

stimulated emission causes a “cascade” of photons

in lab: create inverted populations of atoms

use mirrors to “recycle” stimulating photons

→ this is a laser! light amplification by stimulated emission of radiation

in cosmos: inverted populations of molecules

maser: microwave amplification by stimulated emission of radiation

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need *nonthermal* mechanism to “pump” upper level