Astronomy 501: Radiative Processes

Lecture 9

Sept 17, 2018

Announcements:

- **Problem Set 3** due Friday at start of class

Last time: radiative properties of a two-level system

Q: emission processes? absorption? what do they depend on?

Inverted Populations

two-level absorption coefficient is:

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12} \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) \tag{1}$$

note that the algebraic *sign*

depends on population levels, i.e., on n_2/n_1

normally, lower level more populated: $n_1 > n_2$

If we can arrange or stumble upon a system where

$$\frac{n_1}{g_1} < \frac{n_2}{g_2} \tag{2}$$

i.e., an *inverted population*, then $\alpha_\nu < 0$!

Q: and then what happens to propagating light?

Q: examples?

Q: how might we arrange an inverted population?

Masers

if $\alpha_\nu < 0$, then propagating light has
exponential *increase* in intensity!

stimulated emission causes a "cascade" of photons

in lab: create inverted populations of atoms
use mirrors to "recycle" stimulating photons
$\longrightarrow$ this is a laser! light amplification by stimulated emission of radiation

in cosmos: inverted populations of molecules
maser: microwave amplification by stimulated emission of radiation

how to create inversion?
need *nonthermal* mechanism to "pump" upper level

Scattering

Pure Scattering

Consider an idealized case with radiation propagating through a medium with "*pure scattering*," i.e., scattering, but *no emission*, and *no absorption*

Recall: intensity in a ray is a directional quantity i.e., really $I_\nu = I_\nu(\theta, \phi) = I_\nu(\hat{n})$, with $\hat{n}$ a unit vector toward (θ, ϕ)

in general: scattering preserves photon *number* but *redistributes* both
- photon energy
- photon direction

generally, scattering is different for different incident and scattered angles, i.e., anisotropic

this is generally is (very) non-trivial to calculate but consider even more special case:

- *isotropic* scattering:
  no preferred direction for scattered photons
- photon energy unchanged ("*coherent scattering*")
  good approximation for scattering by non-relativistic e

define **scattering coefficient** $\varsigma_\nu = n_{\rm scat}\sigma_{{\rm scat},\nu}$,
and thus also scattering cross section $\sigma_{\rm scat}$, such that
intensity lost to scattering *out* of ray is

$$dI_\nu = -\varsigma_\nu \; I_\nu \; ds \tag{3}$$

isotropic scattering $\rightarrow$ ς_ν same for all directions

Q: what is intensity scattered into *the ray?*

Isotropic Coherent Scattering

intensity scattered *out* of ray $I_\nu(\hat{n})$ is

$$dI_\nu(\hat{n}) = -\varsigma_\nu \; I_\nu(\hat{n}) \; ds \tag{4}$$

if scattering *isotropic*, the portion *into* $\hat{n}$ is

$$dI_\nu(\hat{n}) = \frac{d\Omega'}{4\pi} \left| dI_\nu(\hat{n}') \right| \tag{5}$$

and so integrating over all possible solid $d\Omega'$ gives

$$dI_\nu(\hat{n}) = \frac{\varsigma_\nu}{4\pi} \int I_\nu \; d\Omega \; ds = \varsigma_\nu \; J_\nu \; ds \tag{6}$$

where J_ν is the angle-averaged intensity and thus for isotropic coherent scattering

$$\frac{dI_\nu(\hat{n})}{ds} = -\varsigma_\nu \left[I_\nu(\hat{n}) - J_\nu \right] \tag{7}$$

and so the source function is

$$S_\nu = J_\nu \tag{8}$$

and the transfer equation can be written

$$\frac{dI_\nu(\hat{n})}{d\tau_\nu} = -I_\nu(\hat{n}) + J_\nu \tag{9}$$

where mean flux $J_\nu = \int I_\nu(\hat{n}')d\Omega'/4\pi$, and $d\tau_\nu = \varsigma_\nu \; ds$

Q: why is this intuitively correct?

Q: what is effect on I_ν of many scattering events?

for coherent, isotropic scattering:

$$\frac{dI_\nu(\hat{n})}{d\tau_\nu} = -I_\nu(\hat{n}) + J_\nu \tag{10}$$

depends on I_ν field in *all directions*

$\Rightarrow$ scattering couples intensity in different directions

if many scattering events, τ_ν large: $I_\nu \to J_\nu$

after large number of mean free paths, photons $\to$ isotropic

$\Rightarrow$ (isotropic) scattering randomizes photon directions
reduces anisotropy

transfer with scattering: integro-differential equation
generally very hard to solve!

Q: transfer equation modification for anisotropic scattering?

Scattering and Random Walks

Can we understand photon propagation with isotropic scattering in a simple physical picture?

simple model: **random walk**
between collisions, photons move in straight "steps"
with random displacement $\widehat{\ell}$
position after N collisions ("steps") is $\vec{r}_N$

idealizations:

- *step length uniform*: $|\widehat{\ell}| = \ell_{\rm mfp}$ mean free path
- *step direction random*: each $\widehat{\ell}$ drawn from isotropic distribution and independent of previous steps
- initial condition: start at center, $\vec{r}_0 = 0$

1. first step $\vec{r}_1 = \hat{\ell}$
length $|\vec{r}_1| = \ell_{\rm mfp}$, direction random
average over ensemble of photons:

- $\langle \vec{r}_1 \rangle = 0$
- $\left\langle r_1^2 \right\rangle = \ell_{\rm mfp}^2$

average positions for *ensemble* of photons is zero
but average distance of *each* photon $\ell_{\rm mfp}$

2. step N has: $\vec{r}_N = \vec{r}_{N-1} + \hat{\ell}$
average over ensemble of photons:
$\langle \vec{r}_N \rangle = \langle \vec{r}_{N-1} \rangle + \left\langle \hat{\ell} \right\rangle = \langle \vec{r}_{N-1} \rangle$
but by recursion

$$\langle \vec{r}_N \rangle = \langle \vec{r}_{N-1} \rangle = \langle \vec{r}_{N-2} \rangle = \ldots = \langle \vec{r}_1 \rangle = 0 \tag{11}$$

$\longrightarrow$ ensemble average of photons displacements still 0
as it must be by symmetry

but what about *mean square* displacement?

$$r_N^2 = \vec{r}_N \cdot \vec{r}_N \quad (12)$$

$$= r_{N-1}^2 + 2\hat{\ell} \cdot \vec{r}_{N-1} + \ell_{\rm mfp}^2 \quad (13)$$

average over photon ensemble

$$\left\langle r_N^2 \right\rangle = \left\langle r_{N-1}^2 \right\rangle + 2 \left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle + \ell_{\rm mfp}^2 \quad (14)$$

Q: what is $\left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle$*?*

$$\left\langle r_N^2 \right\rangle = \left\langle r_{N-1}^2 \right\rangle + 2 \left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle + \ell_{\rm mfp}^2 \tag{15}$$

each photon scattering direction independent from previous

$\left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle = \ell_{\rm mfp} r_{N-1} \left\langle \cos\theta \right\rangle = 0$

so $\left\langle r_N^2 \right\rangle = \left\langle r_{N-1}^2 \right\rangle + \ell_{\rm mfp}^2$

but this means $\left\langle r_N^2 \right\rangle = N \ell_{\rm mfp}^2$

$\rightarrow$ **each** photon goes r.m.s. distance

$$r_{\rm rms} = \sqrt{\left\langle r_N^2 \right\rangle} = \sqrt{N}\ \ell_{\rm mfp} \tag{16}$$

so imagine photons generated at $r = 0$

and, after scattering, are observed at distance L

Q: number N of scatterings if optically thin? thick?

Photon Random Walks and Optical Depth

if travel distance L by random walk
then after N scatterings $L = \sqrt{N}\ \ell_{\rm mfp}$
but photon optical depth is $\tau = L/\ell_{\rm mfp}$
$\rightarrow$ counts number of mean free paths in length L

optically thick: $\tau \gg 1$
many scattering events $\rightarrow$ *this is a random walk!*
$N \stackrel{\rm thick}{\approx} \tau^2$

if *optically thin*: $\tau \ll 1$
scattering probability $1 - e^{-\tau} \approx \tau \ll 1$: *not random walk!*
mean number of scatterings over L is $N \stackrel{\rm thin}{\approx} \tau$

approximate expression good for all τ

$$N \approx \tau + \tau^2 \tag{17}$$

Combined Scattering and Absorption

generally, matter can both scatter and absorb photons
transfer equation must include both
for *coherent isotropic scattering* of *thermal radiation*

$$\frac{dI_\nu}{ds} = -\alpha_\nu (I_\nu - B_\nu) - \varsigma_\nu (I_\nu - J_\nu) \tag{18}$$

giving a source function

$$S_\nu = \frac{\alpha_\nu B_\nu + \varsigma_\nu J_\nu}{\alpha_\nu + \varsigma_\nu} \tag{19}$$

a *weighted average* of the two source functions

thus we can write

$$\frac{dI_\nu}{ds} = -(\alpha_\nu + \varsigma_\nu)(I_\nu - S_\nu) \tag{20}$$

with **extinction coefficient** $\alpha_\nu + \varsigma_\nu$

generalize mean free path:

$$\ell_{\mathrm{mfp},\nu} = \frac{1}{\alpha_\nu + \varsigma_\nu} \tag{21}$$

average distance between photon interactions

in random walk picture:
probability of step ending in *absorption*

$$\epsilon_\nu \equiv \alpha_\nu \ell_{\mathrm{mfp},\nu} = \frac{\alpha_\nu}{\alpha_\nu + \varsigma_\nu} \tag{22}$$

and thus step *scattering probability*

$$\varsigma_\nu \ell_{\mathrm{mfp},\nu} = \frac{\varsigma_\nu}{\alpha_\nu + \varsigma_\nu} = 1 - \epsilon_\nu \tag{23}$$

also known as **single scattering albedo**

source function:

$$S_\nu = \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu \tag{24}$$

Random Walk with Scattering and Absorption

in *infinite medium*: every photon created is eventually absorbed
typical absorption path $\ell_{\mathrm{abs},\nu} = 1/\alpha_\nu$
typical number of scattering events until absorption is

$$N_{\mathrm{scat}} = \frac{\ell_{\mathrm{abs},\nu}}{\ell_{\mathrm{mfp},\nu}} = \frac{\varsigma_\nu + \alpha_\nu}{\alpha_\nu} = \frac{1}{\epsilon_\nu} \tag{25}$$

so typical distance traveled between creation and absorption

$$\ell_* = \sqrt{N_{\mathrm{scat}}}\,\ell_{\mathrm{mfp},\nu} = \sqrt{\ell_{\mathrm{abs},\nu}\ell_{\mathrm{mfp},\nu}} = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \varsigma_\nu)}} \tag{26}$$

diffusion/thermalization length or *effective mean free path*

What about a *finite medium* of size s?
define optical thicknesses $\tau_{\mathrm{scat}} = \varsigma_\nu s$, $\tau_{\mathrm{abs}} = \alpha_\nu\ s$
and $\tau_* = s/\ell_* = \tau_{\mathrm{scat}}^{1/2}(\tau_{\mathrm{scat}} + \tau_{\mathrm{abs}})^{1/2}$

Q: expected behavior if $\tau_* \ll 1$? $\tau_* \gg 1$?

$\tau_* = s/\ell_*$: path in units of photon travel until absorption

$\tau_* \ll 1$: *effectively thin* or *translucent*
photons random walk by scattering, seen before absorption
luminosity of thermal source with volume V is

$$L_\nu \stackrel{\text{thin}}{=} 4\pi\alpha_\nu B_\nu V = 4\pi j_\nu(T)V \tag{27}$$

$\tau_* \gg 1$: *effectively tick*
thermally emitted photons scattered then absorbed before seen
expect $I_\nu \to S_\nu \to B_\nu$
rough estimate of luminosity of thermal source:
most emission from "last scattering" surface of area A
where photons travel $s = \ell_*$

$$L_\nu \stackrel{\text{thick}}{\approx} 4\pi\alpha_\nu B_\nu \ell_* A \approx 4\pi\sqrt{\epsilon_\nu}\, B_\nu A \tag{28}$$

Life Inside a Star

In stars:

- nuclear reactions create energy and γ rays deep in the interior (core)
- the energy and radiation escape to the surface after many interactions

How does this occur?

Consider a point at stellar radius r with temperature $T(r)$ having blackbody radiation at T, and matter

Q: what is intensity pattern (i.e., over solid angle) if T is uniform?
Q: what is the pattern more realistically?
Q: what drives the outward energy flow? what impedes it?
Q: relevant length scale(s) for radiation flow?

if $T(r)$ uniform and has *no gradient*,
so are blackbody intensity B and flux T
$\rightarrow$ no net flow of radiation!

but in real stars: T *decreases with* r

so at r:

- intensity from below greater than from above
- drive net flux outwards
- impeded by scattering and absorption
  on scale $\ell_{\mathrm{mfp},\nu} = (\alpha_\nu + \varsigma_\nu)^{-1}$
- generally $\ell_{\mathrm{mfp},\nu} \ll r$: over this scale, see radiation as
  mostly isotropic with *small dipole*

Radiative Diffusion: Sketch Rosseland Approximation

given a small temperature dipole, expect *net radiation flux*

$$F_\nu^{\text{net}} \sim -\pi \Delta B_\nu \sim -\pi \left[B_\nu(T_{r+\delta r}) - B_\nu(T_{r+\delta r}) \right] \quad (29)$$

$$= -\pi \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial r} \delta r \quad (30)$$

$$= -\pi \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial r} \ell_{\text{mfp},\nu} \quad (31)$$

So the total flux $F = \int F_\nu^{\text{net}} d\nu$ has

$$F = -\frac{4}{3}\pi \frac{\partial_T B}{\alpha_{\text{R}}} \partial_r T \quad (32)$$

- $\vec{F} \propto -\nabla T$: diffusion flux! requires gradient!
- average over ν gives **Rosseland mean absorption coefficient**

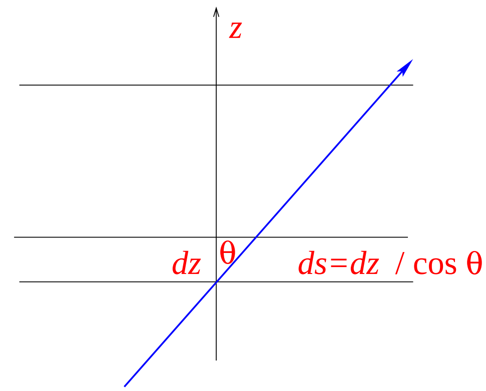
$$\frac{1}{\alpha_{\text{R}}} = \frac{\int (\alpha_\nu + \varsigma_\nu)^{-1} \partial_T B_\nu \, d\nu}{\int \partial_T B_\nu \, d\nu} \quad (33)$$

effective mean free path, weighted by Planck derivative

Director's Cut Extras

Rosseland Approximation in Detail

Imagine a **plane-parallel medium**:
n, ρ, T only depend on z
Think: interior of a star



photon propagation depends only on angle θ
between path direction and $\hat{z}$ *Q: why? why not on* ϕ *too?*

change to variable $\mu = \cos\theta$, and note that
path element $ds = dz/\cos\theta = dz/\mu$, so

$$\mu \frac{\partial I_\nu(z,\mu)}{\partial z} = -(\alpha_\nu + \varsigma_\nu)(I_\nu - S_\nu) \tag{34}$$

note: deep inside a real star like the Sun, $\ell_* \sim 1$ cm $\ll R_\star$
Q: implications?

$\ell_* \sim 1$ cm $\ll R_\star$: rapid thermalization, damping of anisotropy

expect stellar interior to have intensity field that
- changes slowly compared to mean free path
- is nearly isotropic

so to *zeroth order* in ℓ_*, transfer equation

$$I_\nu = S_\nu - \mu \ell_* \frac{\partial I_\nu(z,\mu)}{\partial z} \tag{35}$$

gives

$$I_\nu^{(0)} \approx S_\nu^{(0)}(T) \tag{36}$$

this is angle-independent, so: $J_\nu^{(0)} = S_\nu^{(0)}$ and $I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$

Iterate to get *first-order approximation*

$$I_\nu^{(1)} \approx S_\nu^{(0)} - \mu \ell_* \partial_z I_\nu^{(0)} = B_\nu - \frac{\mu}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \tag{37}$$

what angular pattern does this intensity field have? why?

to first order, intensity pattern

$$I_\nu^{(1)} \approx S_\nu^{(0)} - \mu \ell_* \partial_z I_\nu^{(0)} = B_\nu - \frac{\mu}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \tag{38}$$

i.e., a dominant isotropic component plus
small correction $\propto \mu = \cos\theta$: a *dipole!*
if T decreases with z, then $\partial_z B_\nu < 0$, and so
intensity brighter downwards (looking into hotter region)

use this find **net specific flux along z**

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu) \ \cos\theta \ d\Omega = 2\pi \int_{-1}^{+1} I_\nu^{(1)}(z,\mu) \ \mu \ d\mu \tag{39}$$

only the *anisotropic* piece of $I_\nu^{(0)}$ of survives *Q: why?*

$$\begin{aligned} F_\nu(z) &= -\frac{2\pi}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \int_{-1}^{+1} \mu^2 \ d\mu && (40) \\ &= -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)} \partial_z B_\nu && (41) \end{aligned}$$

net specific flux along z

$$F_\nu(z) = -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)} \partial_z B_\nu = -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)} \partial_T B_\nu \; \partial_z T \tag{42}$$

since $B_\nu = B_\nu(T)$

total integrated flux

$$F(z) = \int F_\nu(z) \; d\nu = -\frac{4\pi}{3} \partial_z T \int (\alpha_\nu + \varsigma_\nu)^{-1} \frac{\partial B_\nu}{\partial T} \; d\nu \tag{43}$$

to make pretty, note that

$$\int \partial_T B_\nu \; d\nu = \partial_T \int B_\nu \; d\nu = \partial_T B(T) = \frac{4\pi\sigma T^3}{\pi} \tag{44}$$

and define **Rosseland mean absorption coefficient**

$$\frac{1}{\alpha_{\mathrm{R}}} = \frac{\int (\alpha_\nu + \varsigma_\nu)^{-1} \partial_T B_\nu \; d\nu}{\int \partial_T B_\nu \; d\nu} \tag{45}$$

average effective mean free path, weighted by Planck derivative