Astro 404

Lecture 9

Sept. 16, 2019

Announcements:

- **Problem Set 3 due Friday**

  TA office hours early this week:

  Tomorrow, Tue Sept 17, noon-1pm

  instructor office hours: Wed 11am-noon or by appt

Last time: began to build a theory of stars

Q: simplifying assumptions for "minimally realistic" model?

Q: what specifically does a model quantify?

Q: density and enclosed mass?

Building Models of Stars

simplifying assumptions:
- isolated star
- non-rotating, non-magnetic
- not mass loss (until the end)

fixed total mass, spherically symmetry $\rightarrow$ only r dependences

to characterize physical state everywhere in star, need:
- **mass density** ρ
- **composition** (which elements)
- **temperature** T
- **pressure** P

dependence of each on r: *radial profile*

enclosed mass defined as
mass inside radius r: mass coordinate $m \leftrightarrow r$

$$m(r) = \int_0^r \rho \; dV = 4\pi \int_0^r \rho(r) \; r^2 \; dr$$

Newtonian Gravitational Field

for *gravitating point* mass M:

- acceleration independent of test mass
- thus only depends on "source" M

formally: can write *test mass force* $\vec{F}_m = m\vec{g}$
and thus in the presence of a gravity source M
  i.e., given the existence and amount *mass*
any and all test particles at point $\vec{r}$ feel acceleration

$$\vec{a} = \vec{g}(\vec{r}) \tag{1}$$

Galileo's Tower of Pisa result; Einstein's equivalence principle
$\Rightarrow$ Newtonian physical interpretation: each mass M sets up
  its own **gravitational field $\vec{g}$** throughout space

Gravity from many sources: Superposition

Thus far: only considered single point masses
what if we add more gravity sources–i.e., more masses?

If one point particle of mass m at $\vec{r}$
gravity is

$$\vec{g} = -\frac{Gm}{r^2}\hat{r} \tag{2}$$

for many particles: use principle of superposition
$\Rightarrow$ take vector sum of gravitational acceleration

bad news: this can be complicated!
good news: spherical symmetry drastically simplifies
best news: you already have the technology in hand
Q: what's that? hint–it was in PHYS 212

Gravitation and Electrostatics: Family Resemblance

how sum up? how do the integral?

You already have the technology! Notice similarity:

	Electrostatics	*Gravity*
"charge"	q	m
force	$qQ/4\pi\epsilon_0 r^2\ \hat{r}$	$-GmM/r^2\ \hat{r}$
field	$\vec{F}_q = q\vec{E}$	$\vec{F}_m = m\vec{g}$

formally identical inverse square law forces!
(except sign, and $\pm q$ allowed, $m \geq 0$)

So: can import electrostatics technology
Memory lane: Gauss' Law from EM
`www: PHYS 212`

Gauss' Law in E&M

consider a point charge Q

enclose in sphere: $\vec{E}$ normal to surface $\vec{S}$

$$\int_S \vec{E} \cdot d\vec{S} = E \int_S dS = \frac{Q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{q}{\epsilon_0} \tag{3}$$

miracle: holds for all $\vec{E}$ and surfaces $\vec{S}$

$$\text{electric flux} = \int_S \vec{E} \cdot d\vec{S} = \frac{q_{\text{enc}}}{\epsilon_0} \tag{4}$$

where q_{enc} is total charge enclosed in surface S

Gauss' Law for gravity: for point mass M

$$\int_S \vec{g} \cdot d\vec{S} = -\frac{GM}{r^2} 4\pi r^2 = -4\pi GM \tag{5}$$

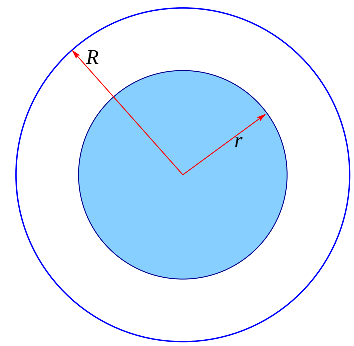
and in general:

$\int_S \vec{g} \cdot d\vec{S} = -4\pi G M_{\text{enc}}$

A Gravitating Sphere

spherical mass distribution: $\rho(r)$ and $\vec{g}(r)$

Gauss' Law: choose spherical surface



$$\int_S \vec{g} \cdot d\vec{S} = 4\pi r^2 g(r) = -4\pi G m(r)$$

where $m(r)$ is *enclosed mass!*

solve:

$$\vec{g}(r) = -\frac{Gm(r)}{r^2}\hat{r}$$

note similarity to point-source formula
but this works for *any* spherical mass distribution
and works inside, outside mass distribution!

Q: field at center?
Q: field if hollow out inside and you're there?
$\Rightarrow$ field is same as if interior mass concentrated at center!

iClicker Poll: Maximal Gravity

imagine the Earth's density were uniform (constant)

Where would the gravitational acceleration be the strongest?

A at the center

B at the surface

C at the Moon's distance

D none of the above

Gravitational Potential

gravitational potential Φ defined as

$$\vec{g} = -\nabla\Phi$$

which gives the potential in terms of the field as

$$\Phi = -\int \vec{g} \cdot d\vec{s}$$

up to a constant – usually set $\Phi \to 0$ as $r \to \infty$

for a spherically symmetric mass distribution the **gravitational potential energy** of a test mass μ is

$$\Omega = \mu\Phi \overset{\text{sph}}{=} -\frac{G\ m(r)\ \mu}{r} \tag{6}$$

Q: significance of minus sign?

Q: how to find gravitational potential energy of entire star?

Gravitational Potential Energy

for a spherically symmetric, continuous matter distribution the **total gravitational potential energy** is (see Extras) sum of contributions at each mass shell dm:

$$\Omega = - \int_0^M \frac{Gm}{r} \, dm \tag{7}$$

where the integration is over the mass coordinate
Q: why the minus sign? physical significance?

result depends on stellar structure via $\rho(r)$ or $m(r)$
Q: order of magnitude for star of mass M and radius R?
Q: result for infinitely thin shell of size R?

Gravitational Potential Energy: Order of Magnitude

order of magnitude estimate from dimension analysis:
given M and R, and universal constant G
only one combination has units of energy: GM^2/R (check it!)

with the correct *minus sign to indicate a bound system*
we expect that

$$\Omega = -\alpha \frac{GM^2}{R} \tag{8}$$

where α is a dimensionless constant

example: an *infinitely thin shell* has

$$\Omega_{\text{shell}} = -\frac{GM^2}{R}$$

and we see that $\alpha_{\text{shell}} = 1$

Stellar Stability I

the Sun's size is constant on human timescales

$\Rightarrow$ not expanding, collapsing

$\Rightarrow$ $\boxed{\text{stable}}$

Why?

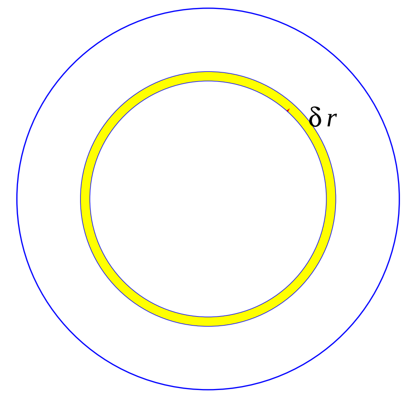
Appreciate: not a trivial result, could have been otherwise

compare terrestrial, interstellar clouds–irregular shape, morph with time

$\rightarrow$ in lab, expect gases expand to fill available space

iClicker Poll: Forces on a Shell of Solar Gas

Consider a shell of gas in the Sun, **at rest**
i.e., Sun not expanding, contracting



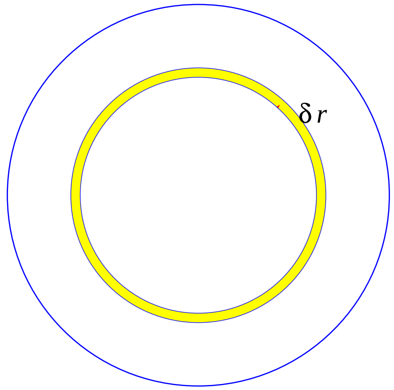
How many forces are acting on this shell?

A zero

B only one

C more than one

Consider a shell of gas in the Sun, **at rest**
radius r, thickness $\delta r \ll r$
shell area $A = 4\pi r^2$
shell volume



$$V = \frac{4\pi}{3}[(r+\delta r)^3 - r^3] \approx 4\pi r^2\,\delta r = A\,\delta r$$

shell mass $m_{\text{shell}} = \rho V = \rho A\ \delta r$

shell weight $F_{\text{w}} = -g m_{\text{shell}} = -g\rho A\ \delta r$:
downward force, but doesn't fall!?

Q: why? gas has weight–why not all at our feet?

Pressurized Stars

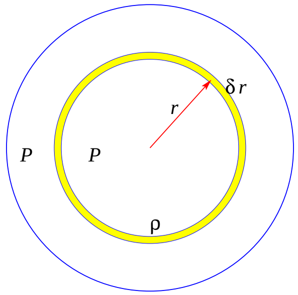
seemingly static nature of star sizes
means surface bulk is at rest
star surface has *zero acceleration* $\rightarrow$ *zero net force*
but gravity definitely present, so another force must exist!

we know stars made of gas, at $T > 0$
so **gas pressure forces** certainly present
$\rightarrow$ a promising candidate to offset gravity

Q: how would this work?
Q: everyday examples that are similar?

iClicker Poll: Pressure in a Star

Consider a star with gas that everywhere has **constant, uniform pressure** P



for a shell of mass $\delta m = \rho\ A\ \delta r$ feeling gravity $g(r)$

What value of P will support this shell in a stable way?

A $P < \delta m\ g/A$

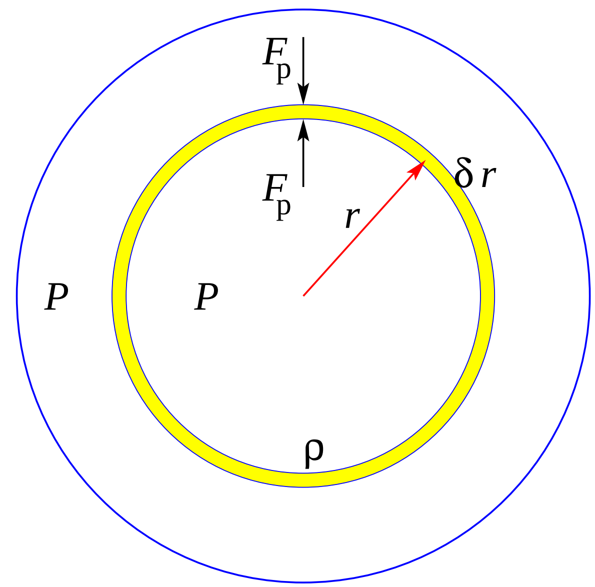
B $P = \delta m\ g/A$

C $P > \delta m\ g/A$

D none of these will support the shell in a stable way

Uniform Pressure Star: Fails Uniformly!

consider a star with *uniform pressure P throughout*



for a *shell of area A*
pressure force from below: $F_{\rm up} = PA$
this acts upward, oppose gravity. Yay!

but: above the shell,
pressure force: $F_{\rm down} = PA$
same magnitude, opposite direction!

in this situation: $F_{\rm up} - F_{\rm down} = (P_{\rm up} - P_{\rm down})A = 0$
the pressure forces cancel!
pressure has no net effect!

under uniform pressure, no net force!

real life example: humans!

atmospheric pressure $P_{\text{atm}} \simeq 10^5 \text{Newton/m}^2$

- force on front of body $\approx 10^5$Newton!
- would send you flying if unbalanced!
- but uniform pressure horizontally $\rightarrow$ forces cancel

yet we know in real life that

pressure can provide support, including against gravity!

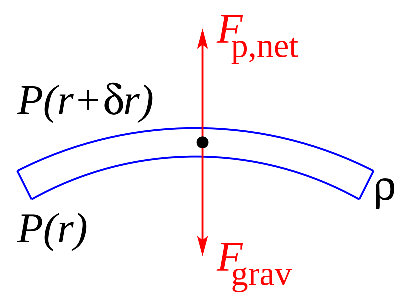
- balloon: inward elastic force vs outward P
- car tire: pressure holds up car's entire weight!

Q: where did we go wrong here? how to fix this?

Nonuniform Pressure: Net Force Emerges

a net pressure force *is* possible
if pressure is *nonuniform* inside the star
that is: $P(r) \neq const$

pressure forces on gas shell at $(r, r + \delta r)$



- from below, upward pressure $P_{\rm up} = P(r)$
- from above $P_{\rm down} = P(r + \delta r)$

net upward force is

$$F_{\rm net,up} = (P_{\rm up} - P_{\rm down})\ A$$
$$= [P(r) - P(r + \delta r)]\ A \quad (9)$$
$$= -\frac{dP}{dr}\ \delta r\ A \quad (10)$$

nonzero if pressure not constant!

Q: why the sign? what does this mean physically?

Q: mathematical condition for a stationary star?

Hydrostatic Equilibrium

net pressure force on shell: $F_{\text{net,up}} = -dP/dr \ \delta r \ A$
upward direction if $dP/dr < 0$: $\rightarrow$ pressure decreases outward, increases inward so on Earth, air is thin at altitude!

if so, star attains **hydrostatic equilibrium:**
$F_{\text{weight}} = F_{\text{pressure}}$

pressure gradient exactly balances downward gravity

$$\frac{dP}{dr} = -g\rho = -\frac{G \ m(r) \ \rho(r)}{r^2}$$

Note what this means:
$\rightarrow$ Sun's mechanical structure $\rho(r), m(r)$ intimately related to thermal structure via pressure profile $P(r)$

Director's Cut Extras

Potential Energy of a Spherical Mass Distribution

For spherical continuous (i.e., with density ρ) mass distribution

$$\frac{d\Phi}{dr} = -\frac{Gm(r)}{r} \tag{11}$$

from this we want to find the gravitational potential energy

For a set of point masses:
the poential energy is the sum over distinct pairs

$$\Omega = -\frac{1}{2}\sum_{i}\sum_{j\neq i}\frac{Gm_i m_j}{r_{ij}} \tag{12}$$

where

- the double sum is over $i, j = 1, \ldots, N$ particles and omits identical pairs $j = i$
- $r_{ij} = |\vec{r}_i - \vec{r}_j|$ is the distance between i and j
- and the factor of $1/2$ corrects for double counting of pairs

generalizing to a smooth distribution of mass, we have

$$\Omega = -\frac{1}{2}\int \rho \; \Phi \; dV \tag{13}$$

for our spherically symmetric case

we can use the mass coordinate $dm = \rho \; dV$:

$$\Omega = -\frac{1}{2}\int \Phi \; dm \tag{14}$$

and then we integrate by parts

$$\Omega = \frac{1}{2}\int m \; d\Phi = \frac{1}{2}\int m \; \frac{d\Phi}{dr} dr \tag{15}$$

and now using the relation above

$$\Omega = -\frac{1}{2}\int \frac{Gm^2}{r^2} dr \tag{16}$$

we have

$$\Omega = -\frac{1}{2}\int \frac{Gm^2}{r^2}dr \tag{17}$$

we integrating by parts again: $\int u\ dv = uv - \int v\ du$

here $u = Gm^2/2$, $dv = -dr/r^2 = d(1/r)$

and the uv term is $Gm^2/2r|_0^\infty = 0$

so we finally have

$$\Omega = -\int \frac{Gm}{r}dm \tag{18}$$

which was to be shewn,

and where the prefactor of 1/2 is canceled by a factor of 2 from the differential of m^2