Astro 404
Lecture 10
Sept. 18, 2019

Announcements:

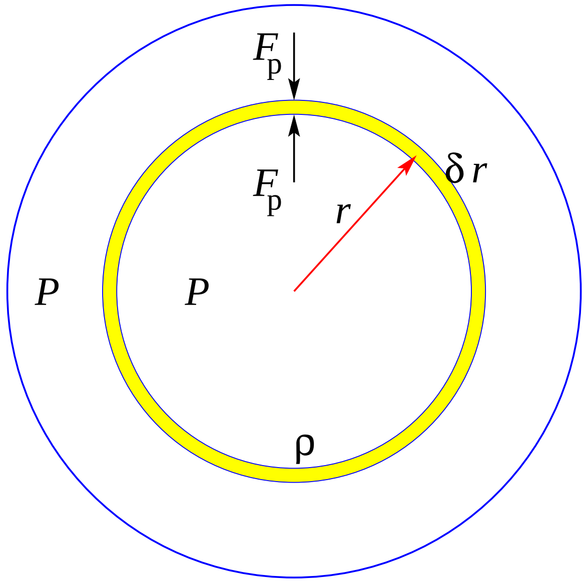
- **Problem Set 3 due Friday**
  instructor office hours: today 11am-noon or by appt

Last time: stars as **self-gravitating spheres**
Q: gravity field of a sphere at r?
Q: order of magnitude of gravitational potential energy for M, R?
Q: what keeps the Sun from collapsing to a black hole?

Uniform Pressure Star: Fails Uniformly!

consider a star with *uniform pressure P throughout*

for a *shell of area A*
pressure force from below: $F_{\text{up}} = PA$
this acts upward, oppose gravity. Yay!



but: above the shell,
pressure force: $F_{\text{down}} = PA$
same magnitude, opposite direction!

in this situation: $F_{\text{up}} - F_{\text{down}} = (P_{\text{up}} - P_{\text{down}})A = 0$
the pressure forces cancel!
pressure has no net effect!

under uniform pressure, no net force!

real life example: humans!

atmospheric pressure $P_{\text{atm}} \simeq 10^5$ Newton/m$^2 = 10^5$ Pa

- force on front of body $\approx 10^5$ Newton!
- would send you flying if unbalanced!
- but uniform pressure horizontally $\rightarrow$ forces cancel

yet we know in real life that

pressure can provide support, including against gravity!

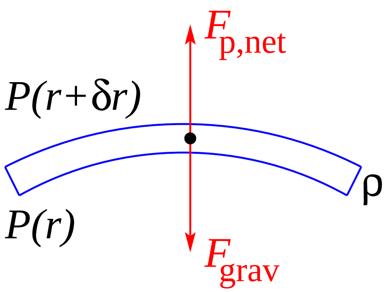
- balloon: inward elastic force vs outward P
- car tire: pressure holds up car's entire weight!

Q: where did we go wrong here? how to fix this?

Nonuniform Pressure: Net Force Emerges

a net pressure force *is* possible
if pressure is *nonuniform* inside the star
that is: $P(r) \neq const$

pressure forces on gas shell at $(r, r + \delta r)$



- from below, upward pressure $P_{\mathrm{up}} = P(r)$
- from above $P_{\mathrm{down}} = P(r + \delta r)$

net upward force is

$$F_{\mathrm{net,up}} = (P_{\mathrm{up}} - P_{\mathrm{down}})\ A$$

$$= [P(r) - P(r + \delta r)]\ A \tag{1}$$

$$= -\frac{dP}{dr}\ \delta r\ A \tag{2}$$

nonzero if pressure not constant!

Q: why the sign? what does this mean physically?

Q: mathematical condition for a stationary star?

Hydrostatic Equilibrium

net pressure force on shell:

$$F_{\text{net,up}} = -dP/dr \ \delta r \ A \tag{3}$$

upward direction if $dP/dr < 0$: $\quad \rightarrow$ pressure decreases outward, increases inward so on Earth, air is thin at altitude!

net gravity force = weight of shell:

$$F_{\text{weight}} = \delta m \ g(r) = \rho(r) \ g(r) \ A\delta r \tag{4}$$

when forces balance: star attains hydrostatic equilibrium: pressure gradient exactly balances downward gravity for every mass shell in star:

$$F_{\text{pressure}} = F_{\text{weight}} \tag{5}$$

$$-\frac{dP}{dr} \ A \ \delta r = \rho(r) \ g(r) \ A \ \delta r \tag{6}$$

shell volume cancels!

The Mighty Equation of Hydrostatic Equilibrium

for a spherical star in hydrostatic equilibrium

$$\frac{dP}{dr} = -g\rho = -\frac{G\ m(r)\ \rho(r)}{r^2}$$

Lesson:

a star's mechanical structure $\rho(r), m(r)$ intimately related to thermal structure via pressure profile $P(r)$

Q: how to solve for $P(r)$? boundary conditions?

Stellar Pressure

hydrostatic equilibrium: pressure gradient balances gravity

$$\frac{dP}{dr} = -\frac{G\ m(r)\ \rho(r)}{r^2} \tag{7}$$

integrate to solve for pressure

$$\begin{aligned} \int_0^r \frac{dP}{dr} dr &= P(r) - P(0) \qquad (8) \\ &= -\int_0^r \frac{G\ m(r)\ \rho(r)}{r^2} dr \qquad (9) \\ & \qquad (10) \end{aligned}$$

integration requires *boundary conditions*

- $P(0) = P_c$ pressure at center: **central pressure**
- $P(R)$ pressure at surface

iClicker Poll: Surface Pressure

Consider a star, mass M and radius R, in hydrostatic equilibrium

What what is pressure $P(R)$ at surface boundary?

A $P(R) < 0$

B $P(R) = 0$

C $P(R) > 0$

D none of the above

Pressure at the Extremes

if star fully in hydrostatic equilibrium
pressure gradient balances gravity: $dP/dr = -Gm\rho/r^2$

- outer boundary defined by $\rho(R) = 0$
- so there, $dP/dr = 0$: pressure minimized $\rightarrow$ $P(R) = 0$

thus the integration

$$P(R) - P(0) = -\int_0^R \frac{G\ m(r)\ \rho(r)}{r^2} dr \tag{11}$$

gives the *central pressure*

$$P_{\rm c} = P(0) = \int_0^R \frac{G\ m(r)\ \rho(r)}{r^2} dr \tag{12}$$

and thus

$$P(r) = P_c - \int_0^r \frac{G\ m(r)\ \rho(r)}{r^2} dr \tag{13}$$

pressure drops monotonically from the central value

Hydrostatic Equilibrium: Mass Coordinate Picture

recall we can label star interior via radius

but also by *mass coordinate* $dm = 4\pi r^2 \ \rho \ dr$

where $m \in [0, M]$, and where $r(m)$ varies for different density profiles ρ

in these coordinates:

$$\frac{dP}{dr} = \frac{dP}{dm} \ \frac{dm}{dr} = 4\pi r^2 \frac{dP}{dm} \tag{14}$$

and so hydrostatic equilibrium $dP/dr = -G\rho m/r^2$ gives

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{15}$$

$$P = P_c - \int_0^M \frac{Gm \ dm}{4\pi r^4} \tag{16}$$

Q: order of magnitude of central pressure P_c?

Central Pressure: Order of Magnitude

order of magnitude:

- find *scaling* with parameters M, R, and constants like G
- ignore dimensionless constants like 2, π, etc

estimate *central pressure:*

$$P_c \sim \frac{\text{characteristic force}}{\text{characteristic area}} \quad (17)$$

$$\sim \frac{M\, g(R)}{R^2} = \frac{GM^2/R^2}{R^2} = \frac{GM^2}{R^4} \quad (18)$$

or try characteristic energy GM^2/R per volume R^3: same answer!

or estimate from integral: same answer

$$P_c = \frac{1}{4\pi}\int_0^M \frac{Gm\ dm}{r^4} \sim \frac{GM^2}{R^4} \quad (19)$$

Q: is the answer physically reasonable–scaling with M, R?

Limits to Pressure

central pressure exact expression

$$P_{\mathrm{c}} = \frac{1}{4\pi} \int_0^M \frac{Gm\ dm}{r^4} \tag{20}$$

note denominator $r^4 < R^4$; use this to set a *strict lower bound*

$$P_{\mathrm{c}} > P_{\mathrm{c,min}} = \frac{1}{4\pi} \int_0^M \frac{Gm\ dm}{R^4} = \frac{GM^2}{8\pi R^4} \tag{21}$$

order of magnitude estimate with $1/8\pi$ factor

Q: for which stars is P_c smallest? largest?

Central Pressure of Stars

plug in numbers for mass and radius:

$$P_{\text{c,min}} \overset{\text{Sun}}{=} 5 \times 10^{13}\ \text{N/m}^2 = 4 \times 10^8\ \text{atm}$$
$$\overset{\text{Sirius B}}{=} 9 \times 10^{21}\ \text{N/m}^2 = 10^{17}\ \text{atm white dwarf}$$

- solar central pressure huge!
- white dwarfs have similar mass but much smaller billions of times larger still!
- giants and supergiants: limit much smaller

iClicker Poll: Thermal Physics and You

Vote your conscience!

What thermal physics have you seen?

A thermal physics? haven't had the pleasure at all

B yes for ideal gases, but not much more

C I'm a PHYS 213 alum. But don't remind me.

D I'm a PHYS 213 alum. Looooved it!

E I'm a PHYS 427 alum – thermal Jedi!

Ideal Gases

ideal gas: free particles in thermal equilibrium
particles in constant random motion in all directions

ideal gas pressure P, temperature T, and density ρ
not all independent: related by **ideal gas equation of state**

$$PV = N\,kT \tag{22}$$

for a fluid element with volume V and number N of particles
note: in physics and astronomy, N counts particles one by one
but chemists count in units of moles, which gives $PV = \mathcal{N}_{\text{mole}}\mathcal{R}T$

k: Boltzmann's constant

Ideal Gas Equation of State

thus we can write

$$P = n\,kT \tag{23}$$

with n the *number density*:
(number per unit volume) of gas particles

Q: how is this related to the mass density ρ of the gas?

Ideal Gases: Mass and Energy Densities

for gas fluid element of mass M, with particle number N
mass density

$$\rho = \frac{M}{V} = \frac{M}{N}\frac{N}{V} = m_{\rm g} n \qquad (24)$$

where $m_{\rm g}$ is *average mass of one gas particle*

so ideal gas equation of state is

$$P = n\ kT = \frac{\rho\, kT}{m_{\rm g}} \qquad (25)$$

so pressure depends on **both** density and temperature: $P \propto \rho T$

Q: what about ideal gas internal energy–form? value?