Astro 404

Lecture 11

Sept. 20, 2019

Announcements:

- **Problem Set 3 due online 5pm today**
- **Problem Set 4 due next Friday**

Last time:

- hydrostatic equilibrium

Q: why hydro? why static? why equilibrium?

Q: what's the mighty equation?

Q: what would the Sun do if gravity switched off ($G = 0$)?

- ideal gas equation of state

Q: mighty equation? in terms of mass density ρ?

Ideal Gas: Macro vs Micro Pictures

gases on microscopic scale
a swarm of particles, for example atoms or molecules

- gas particles have *empty space between them*
  not packed together as in liquid or solid

- gas particles are in *constant random motion*
  act as free particles (constant velocity) between collisions

- *collide elastically with each other, container walls* (if any)
  exchange energy & momentum $\longrightarrow$ distribution of speeds

on macroscopic scales (i.e., how we see things)
particle motions perceived as *temperature*

iClicker Poll: Gas Particle Speeds

consider a parcel of gas:

• macroscopically, gas is at rest (not moving/blowing)

• at room temperature T

in this gas:

the average particle velocity $\vec{v}$ and speed $v = |\vec{v}|$ are:

A $\vec{v} = 0$ and $v = 0$

B $\vec{v} = 0$ and $v > 0$

C $\vec{v} \neq 0$ and $v = 0$

D $\vec{v} \neq 0$ and $v > 0$

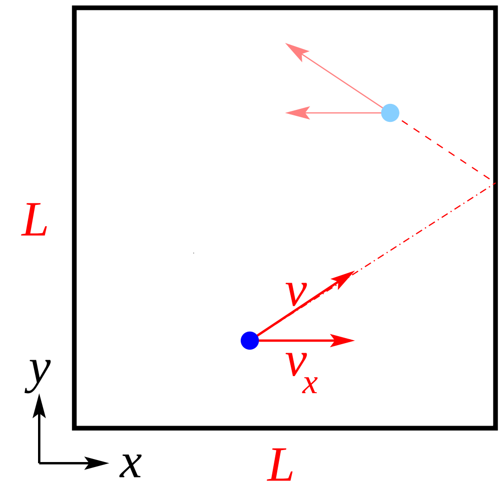
average particle velocity vector vanishes: $\langle \vec{v} \rangle = 0$
why? *not* because particles are still
rather: equal numbers with $v_x > 0$ vs $v_x < 0$ $\rightarrow$ averages to zero
otherwise: gas would have net v_x, wouldn't be at rest!

note microscopic–macroscopic (particle–bulk) correspondence:
micro: equal probabilities for particle $\vec{v} > 0$ and $\vec{v} < 0$
macro: corresponds to bulk gas speed $\vec{u}_{\text{gas}} = 0$

since particles are moving, speeds $\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle > 0$
$\rightarrow$ average kinetic energy of each gas particle is nonzero

Ideal Gas: Microscopic Picture

consider ideal gas of particles with mass m_{g}
in cubic box of size L and volume $V = L^3$



a particle of speed v_x bounces between yz walls

- bounce return time for same wall: $\delta t = 2L/v_x$
- bounces are *elastic*: $v_x^{\mathrm{after}} = -v_x^{\mathrm{before}}$
  so $\delta v_x = v_x^{\mathrm{after}} - v_x^{\mathrm{before}} = 2v_x$, and $\delta p_x = 2m_{\mathrm{g}} v_x$

so collision changes particle momentum!
this exerts *force on wall*

$$F_x = \frac{\delta p_x}{\delta t} = \frac{2m_{\mathrm{g}} v_x}{2L/v_x} = \frac{m_{\mathrm{g}} v_x^2}{L} \qquad (1)$$

Q: force per unit area in single particle collision?
Q: total force per area due to N particles?

for particle of mass $m_{\rm g}$ and speed v_x:
force per unit area of wall

$$\frac{F_x}{A} = \frac{F_x}{L^2} = \frac{m_{\rm g} v_x^2}{L^3} = \frac{m_{\rm g} v_x^2}{V} \tag{2}$$

total force per area: *pressure!*
sums over all N particles

$$P = \sum_{i=1}^{N} \frac{m_{\rm g} v_{i,x}^2}{V} = N \frac{m_{\rm g} \langle v_x^2 \rangle}{V} \tag{3}$$

where $\langle v_x^2 \rangle$ means *average v_x^2 over all particles*

gas speed distribution is *isotropic* = same in all directions
Q: what does this means for $\langle v_x^2 \rangle$ vs v_y^2? for $v_{\rm tot}^2$?

Ideal Gas: Equation of State Emerges!

if gas speeds are *isotropic = no preferred direction*, then

- $\langle v_x^2 \rangle = \langle v_x^2 \rangle + \langle v_z^2 \rangle$
- $\langle v^2 \rangle = \langle v_x^2 + v_y^2 + v_z^3 \rangle = 3\langle v_x^2 \rangle$

and so the pressure is

$$P = N\frac{m_{\rm g}\langle v^2 \rangle/3}{V} \tag{4}$$

woo hoo! our microscopic collision theory shows

- $P \propto 1/V$ check! *Q: microscopic physical reason?*
- $P \propto N$ check! *Q: microscopic physical reason?*

to fully match ideal gas equation of state
what does this mean about temperature?

microscopic theory gives ideal gas pressure

$$P = N\frac{m_{\rm g}\langle v^2\rangle/3}{V} \tag{5}$$

- $P \propto N$: more particles $\rightarrow$ collisions more frequent
- $P \propto 1/V$: more V $\rightarrow$ collisions less frequent

and so to match ideal gas law $P = N\,kT/V$ we find

$$kT = \frac{1}{3}m_{\rm g}\langle v^2\rangle \tag{6}$$

temperature proportional to square of average particle speed

typical gas particle speed (root-mean-square):

$$v_{\rm rms} = \sqrt{\frac{3kT}{m_{\rm g}}} \tag{7}$$

hotter gas $\leftrightarrow$ faster particles

Ideal Gas Internal Energy

internal energy sum all gas particle
• kinetic energy, plus
• rotational energy if any

for a *monatomic* ideal gas (non-rotating particles)
with N particles at temperature T
non-relativistic internal energy is

$$U = \frac{1}{2} N m \langle v^2 \rangle = \frac{3}{2} N \; kT \qquad (8)$$

ideal internal energy only depends on temperature T
internal energy also known as *"thermal energy"*

Q: what is energy per unit volume?

Ideal Gas Energy Density

ideal gas internal energy

$$U = \frac{3}{2}N\ kT = \frac{3}{2}PV \tag{9}$$

so ideal gas *inter energy per unit volume*

or **internal energy density**, for monatomic gas

$$\varepsilon = \frac{U}{V} = \frac{3}{2}n\ kT = \frac{3}{2}P \tag{10}$$

will be useful to define internal energy per unit mass

$$u = \frac{U}{N m_{\mathrm{g}}} = \frac{\varepsilon}{\rho} = \frac{3}{2}\frac{kT}{m_{\mathrm{g}}} \tag{11}$$

Stars Average Temperature

a star's *total internal energy*

$$U = \int \varepsilon \; dV = \int \frac{\varepsilon}{\rho} \rho \; dV = \int u \; dm = \langle u \rangle M \tag{12}$$

where $\langle u \rangle$ is average internal energy per unit mass

but for ideal gas, $u = 3/2 \; P/\rho = 3kT/2m_{\rm g}$, so

$$U = \frac{3}{2} \frac{M}{m_{\rm g}} \langle kT \rangle \tag{13}$$

and thus **average temperature** is

$$\langle kT \rangle = \frac{2}{3} \frac{m_{\rm g} U}{M} \tag{14}$$

Stellar Energy Budget: Equilibrium

hydrostatic equilibrium condition

$$dP = -\frac{G\ m(r)}{r^2}\rho(r)\ dr \tag{15}$$

multiply by $V = 4pi/3\ r^3$ and integrate

left side:

$$\int_0^R V\ dP = [PV]_0^R - \int_0^R P\ dV \tag{16}$$

but $[PV]_0^R$ *Q: why?*, so we have

$$\int_0^R V\ dP = -\int_0^R P\ dV = -\frac{2}{3}\int_0^R \varepsilon\ dV = -\frac{2}{3}\int_0^R \varepsilon\ dV = -\frac{2}{3}U \tag{17}$$

righthand side:

$$-\frac{1}{3}\int \frac{Gm}{r}\rho\ 4\pi r^2\ dr = -\frac{1}{3}\int \frac{Gm\ dm}{r} = \frac{1}{3}\Omega \tag{18}$$

The Virial Theorem

for an ideal gas in hydrostatic equilibrium:

$$U = -\frac{1}{2}\Omega \qquad (19)$$

internal energy is minus half gravitational potential energy:
the Virial theorem

this is a powerful tool for understanding stellar equilibria

Virial application I: average stellar temperature

$$\langle kT \rangle = \frac{2}{3}\frac{m_{\rm g} U}{M} = \frac{1}{3}\frac{m_{\rm g}\Omega}{M} \qquad (20)$$

$$= \frac{1}{3}\frac{m_{\rm g}}{M}\int \frac{Gm\ dm}{r} \qquad (21)$$

Q: order of magnitude for star of mass M, radius R?

Average Temperature

to order of magnitude,

$$\langle kT \rangle = \frac{1}{3}\frac{m_{\mathrm{g}}}{M}\int \frac{Gm\ dm}{r} \sim \frac{GMm_{\mathrm{g}}}{R} \tag{22}$$

Q: check–reasonable dependence on M? on R?

plug in numbers for the Sun:

$$\langle kT_{\odot} \rangle \sim 10^3\ \mathrm{eV} = 1\ \mathrm{keV} \tag{23}$$

$$\langle T_{\odot} \rangle \sim 10^7\ \mathrm{K} = 10\ \mathrm{MK} \tag{24}$$

where eV = electron volt = $e \cdot 1$ Volt = 1.602×10^{-19} Joule

Q: comparison with surface temperature?

Q: implications?

Lessons: Stellar Interior vs Surface Temperature

for the Sun: $\langle T_\odot \rangle \sim 10\ MK$

compare to surface effective $T_{\rm eff} \approx 5800$ K

- *surface $T_{\rm eff}$ not representative of stellar average!*
- *stellar interiors much hotter than surface!*

in energy units: temperature $kT \gg 1$ eV atomic binding energies

atoms are unbound inside of stars

$\longrightarrow$ *most of stellar interiors are ionized plasma!*

Virial Theorem Application II: Total Energy

Virial theorem relates

- **gravitational potential energy** Ω, and
- **internal energy** U

thus the **total energy** in a star under hydrostatic equilibrium is

$$E_{\rm tot} = U + \Omega = \frac{1}{2}\Omega = -\frac{1}{2}\int \frac{Gm\ dm}{r} \tag{25}$$

Q: sign of Ω? sign of $E_{\rm tot}$? significance?

Q: what if system loses energy (spoiler: radiates) ?

Implications of the Virial Theorem

Virial theorem: *in equilibrium*

gravitational potential energy and internal energy are *related*

$$E_{\text{tot}} = U + \Omega = \frac{1}{2}\Omega \tag{26}$$

- $\Omega = -\int Gm/, dm/r < 0$: *gravitational binding*
- $U = -\Omega/2$: *more bound* $\rightarrow$ *more internal energy*
- $E_{\text{tot}} = \Omega/2 = -U < 0$: *system is gravitationally bound*
  i.e., must supply energy to unbind and move to $r \rightarrow \infty$

if system loses energy while keeping in equilibrium

E_{tot} gets smaller $\rightarrow$ more negative, and so

- $|\Omega|$ larger *Q: and so?*
- U larger $\rightarrow$ more internal energy *Q: why is this bizarre?*

a star in hydrostatic equilibrium that gradually *loses energy*:

- must increase $|\Omega| \sim GM^2/R$
  $\rightarrow$ must become *more compact!*

- also must increase internal energy U
  but $\langle T \rangle \propto U$: star must get hotter!

stars get hotter when they lose energy!?!
stars have "negative specific heat"

but not so strange:
if stars become more compact, gravity stronger
so must heat to increase pressure and maintain hydrostatic equilb.

Ideal Gas: Ultra-Relativistic Case

thus far implicitly assumed *gas speeds are non-relativistic*

- speeds $v \ll c$ and $p \ll m_{\rm g} c$
- so $p^2/m_{\rm g} \ll m_{\rm g} c^2$ and thus $kT \ll m_{\rm g} c^2$

now consider opposite limit: **ultra-relativistic** particles

- $v \approx c$ or even $v = c$ *Q: examples?*
- relativistic momentum $p \gg m_{\rm g} c$
- energy $E = \sqrt{(cp)^2 + (m_{\rm g} c^2)^2} \approx cp$

revisit microscopic pressure derivation

$$P_{\rm ur} = \frac{N \langle pv \rangle}{3V} \approx \frac{1}{3} \frac{N \langle E \rangle}{V} = \frac{1}{3} \frac{U_{\rm ur}}{V} \tag{27}$$

and thus for a relativistic gas $PV = U/3$, and energy density is

$$\varepsilon_{\rm ur} = 3 P_{\rm ur} \tag{28}$$