Astro 404
Lecture 20
Oct. 11, 2019

Announcements:

- **Problem Set 6 due today at 5pm**
  Office Hours: today after class, or by appointment
  yesterday small typos fixed in 2(b) and 2(c)
- Good news: *no homework due next Friday!*
- *Bad news:* **Hour Exam Friday Oct 18.** Info on Compass
- Good news: HW grading has accelerated, and solutions posted

Last time:
energy generation profile due to nuclear reactions
Q: what is enclosed luminosity $l(r)$? local energy flux $F(r)$?
energy transport by radiation
Q: how do photons get out of the Sun?
Q: how do they "know" where to go to leave?

energy generation by nuclear reactions
sets local luminosity (power) density $\mathcal{L}(\rho, T)$
summing (integrating) over volume gives enclosed luminosity

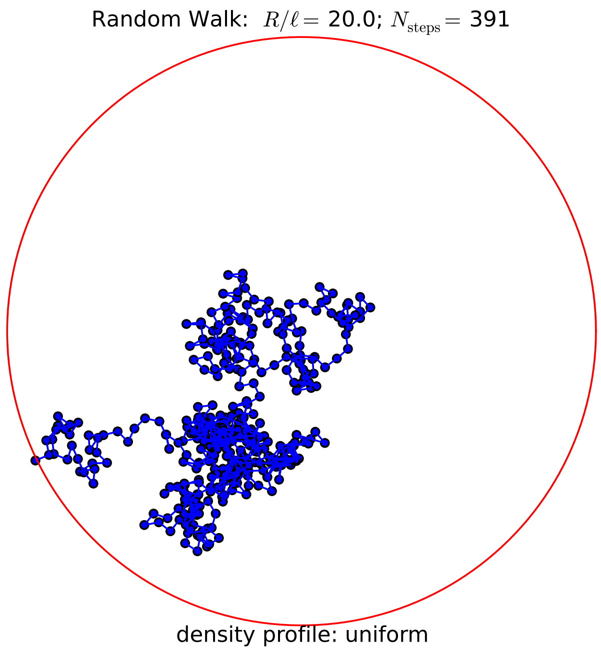
$$l(r) = \int_0^r \mathcal{L} \; dV = 4\pi \int_0^r r^2 \; \mathcal{L}(r) \; dr \tag{1}$$

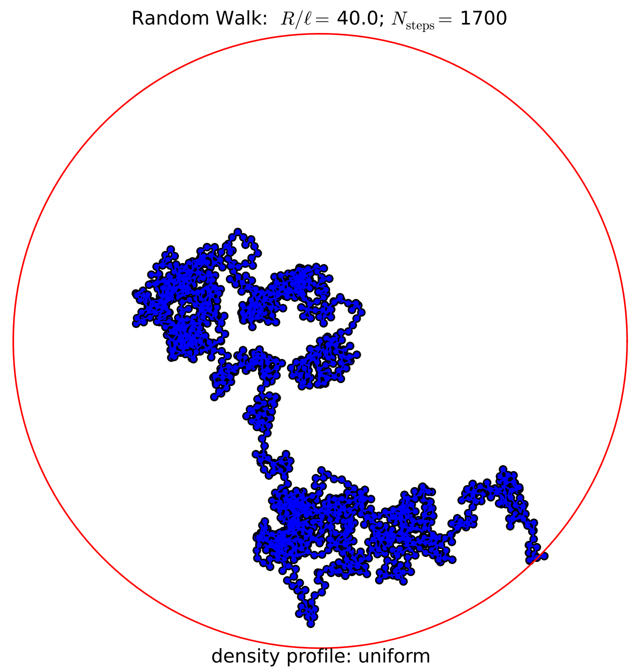
in Sun interior: photons scatter repeatedly: "**random walk**"
on average, each step has equal chance
of "heads" $+\ell_{\mathrm{mfp}}$ and "tails" $-\ell_{\mathrm{mfp}}$
so if *flip many coins for one step, averages to zero*

but if *flip one coin many times*, usually develop
random excess of heads over tails, or vice versa
which means *net progress away from origin!*

when net displacement gets to edge of star, escape!

Effect of Mean Free Path Size





Photon Mean Free Paths

photon mean free path $\ell_{\rm mfp} = 1/n_{\rm sc}\sigma_{\rm sc}$
where $n_{\rm sc}$ is the number density of scatters
and $\sigma_{\rm sc}$ is photon scattering cross section

recall that number and mass densities related by
$\rho_{\rm sc} = m_{\rm sc} n_{\rm sc}$, with scatterer mass $m_{\rm sc}$
so useful to define **opacity**

$$\kappa = \frac{\sigma_{\rm sc}}{m_{\rm sc}} \tag{2}$$

measures cross section per unit scatter mass, and

$$\ell_{\rm mfp} = \frac{1}{n_{\rm sc}\sigma_{\rm sc}} = \frac{1}{\kappa \rho_{\rm sc}} \tag{3}$$

iClicker Poll: Mean Free Paths in Stars

consider photons in the real Sun

How does photon mean free path change in Sun?

A $\ell_{\rm mfp}$ longest in Sun's center, shortest at surface

B $\ell_{\rm mfp}$ shortest in Sun's center, longest at surface

C $\ell_{\rm mfp}$ is uniform in the Sun photon mean free path is

$$\ell_{\rm mfp} = \frac{1}{n_{\rm sc}\sigma_{\rm sc}} = \frac{1}{\kappa\rho_{\rm sc}} \quad (4)$$

in Sun:

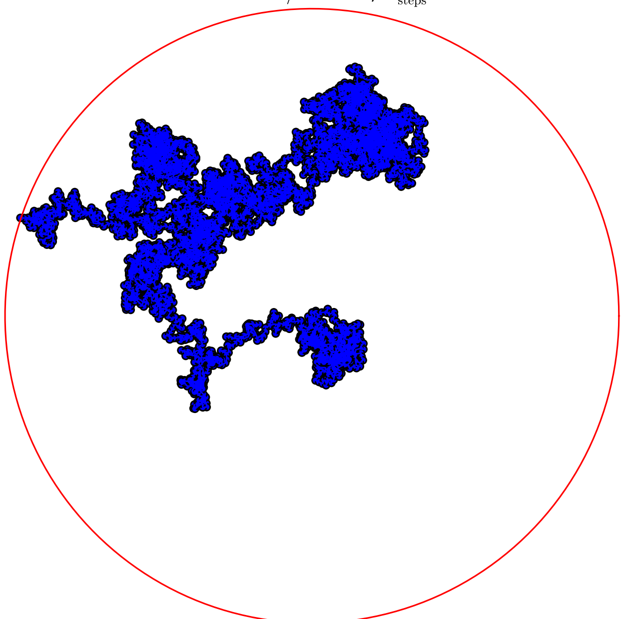
- $\rho(r)$ decreases from center to surface
- and in addition sometimes κ also deceases towards surface

so: mean free path goes from short to long

solar "fog" thins as we go out

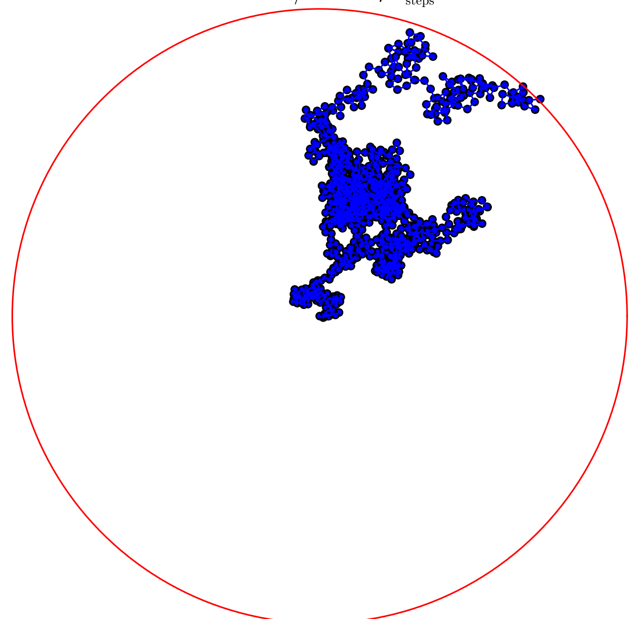
Effect of Density Gradient

Random Walk: $R/\ell = 80.0$; $N_{\rm steps} = 7556$



density profile: uniform

Random Walk: $R/\ell = 80.0$; $N_{\rm steps} = 1258$



density profile: linear

uniform density $\rho(r) = \rho_0$

linear dropoff $\rho(r) = \rho_c(1 - r/R)$

The Sharp-Edged Sun

our *Sun is a gas*, density smoothly drops with radius
it does not really have a surface!

yet it does show a *sharp edge* in images
`www: real-time solar images`

Q: how's that? what is the surface really?

∞

The Solar Photosphere

the surface of the Sun appears sharp
despite random scattering of photons and smooth density profile

photons we see are **not scattered** between Sun and us
and so originate from *final scattering* events in Sun

apparent edge of Sun is **surface of last scattering**
also know as the solar **photosphere**

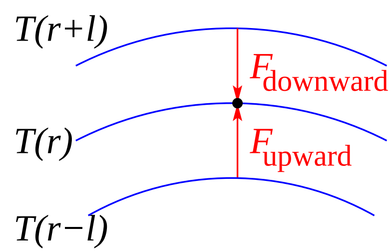
sharpness of photosphere must mean:

- density drops very rapidly near apparent surface
  and thus so does pressure and temperature
- outermost layers are solar "atmosphere"
- where mean free path changes rapidly from short to long
  until $\ell_{\text{mfp}} > R_\odot$: *escape!*

Random Walks and Heat Flow

a random walk is the microscopic picture of **diffusion**
where particles and energy move
from higher concentration to lower due to collisions
note: radiative heat flow **not** due to gas motion!
gas fluid remains at rest! photons scatter through it
later we will discover conditions when flow *is* due to bulk gas motions

for some radius r inside star
consider *energy or heat flux*
from one step above and below

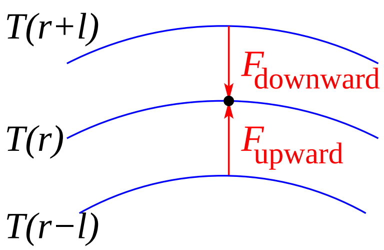


Q: thermal energy flux at temperature T?
Q: if uniform $T(r)$, photon energy flux above? below? net?
Q: what conditions needed to drive heat flow? flow direction?
Q: how do stars satisfy this condition? hint–think globally!
Q: what is size of "one step"?

Temperature Gradients Drive Energy/Heat Flow

thermal radiation has *blackbody* flux $F = \sigma_{SB} T^4$
where σ_{SB} is Stefan-Boltzmann constant, not cross section!

if temperature uniform $T(r) = T_0$:
flux $F = \sigma_{SB} T_0^4$ upward
same as flux downward
no net flow of photons or energy!



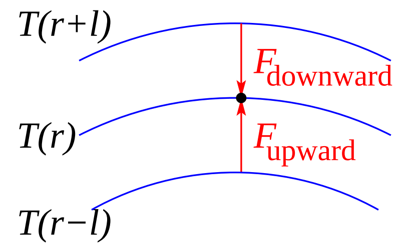
lesson: to create *net flow of photons and energy*
requires *temperature differences with r*: $T(r)$ **gradient!**
and flow direction is from hot $\rightarrow$ cold!

in stars
- *photons and heat produced in the core: kept hot!*
- *photosphere exposed to space: kept cold!*

guarantees temperature differences (gradients) $\rightarrow$ drive heat flow in random walk, "*step*" is *mean free path*

$\ell_{\text{mfp}} = 1/n\sigma = 1/\rho\kappa$

compare flux one step above and below r:



$$F_{\text{downward}} = \sigma_{\text{SB}}\, T^4(r + \ell_{\text{mfp}}) \qquad F_{\text{upward}} = \sigma_{\text{SB}}\, T^4(r - \ell_{\text{mfp}})$$

so *net flux* is the *difference*

$$F_{\text{net}} = F_{\text{up}} - F_{\text{down}} = \sigma_{\text{SB}} \left[T^4(r - \ell_{\text{mfp}}) - T^4(r + \ell_{\text{mfp}})\right]$$

PS6 showed: ℓ_{mfp} small, so do Taylor expansion

$T^4(r + \ell_{\text{mfp}}) \approx T^4(r) + 4\ell_{\text{mfp}}\, T^3(r)\, dT/dr$, which gives

$$F_{\text{net}} = -8\sigma_{\text{SB}}\, \ell_{\text{mfp}}\, T^3(r)\, \frac{dT}{dr} \tag{5}$$

net energy flux depends on

- *temperature gradient* dT/dr
- *mean free path* ℓ_{mfp}

Q: but what determines net flux in the first place?

energy source at core is nuclear reactions
which determines *enclosed luminosity* $l(r)$
which also sets local net energy flux

$$F_{\rm net}(r) = \frac{l(r)}{4\pi r^2} \tag{6}$$

this dictates the needed temperature gradient!

$$\frac{l(r)}{4\pi r^2} = -8\sigma_{\rm SB}\ \ell_{\rm mfp}\ T(r)^3\ \frac{dT}{dr} \tag{7}$$

and finally we can solve for temperature change (i.e., gradient)
and add the correct numerical factors

$$\frac{dT}{dr} = -\frac{3}{16}\frac{\ell_{\rm mfp}}{\sigma_{\rm SB}T(r)^3}\ \frac{l(r)}{4\pi r^2} = -\frac{3}{16}\frac{\kappa(r)\ \rho(r)}{\sigma_{\rm SB}T(r)^3}\ \frac{l(r)}{4\pi r^2} \tag{8}$$

Q: physical story told by this equation?

Equation of Energy Conservation

final equation of stellar structure:

$$\frac{dT}{dr} = -\frac{3}{16}\frac{\ell_{\rm mfp}}{\sigma_{\rm SB} T(r)^3}\ \frac{l(r)}{4\pi r^2} = -\frac{3}{16}\frac{\kappa(r)\ \rho(r)}{\sigma_{\rm SB} T(r)^3}\ \frac{l(r)}{4\pi r^2} \qquad (9)$$

physical content

- radiative heat flux driven by T gradient is set by enclosed luminosity due to nuke reactions
- that is: *energy outflow balances energy creation!* expresses energy conservation!
- note role of mean free path and thus opacity

also note close similarity to derivation and meaning of hydrostatic equilibrium:
outward pressure gradient exactly balances inward gravity to achieve : expression of force balance!

Equations of Stellar Structure

density determines **enclosed mass** (*mass conservation*)

$$\frac{dm}{dr} = 4\pi r^2\ \rho(r)$$

density and and enclosed mass determine **pressure**
due to hydrostatic equilibrium (force balance)

$$\frac{dP}{dr} = -\frac{Gm(r)\ \rho(r)}{r^2}$$

density and temperature determine **nuclear reaction rates**
and thus determine *luminosity* (energy conservation)

$$\frac{dl}{dr} = 4\pi r^2\ \mathcal{L}(\rho, T) = 4\pi r^2\ q(r)\ \rho(r)$$

luminosity, temperature, opacity set **photon diffusion**
and determine *temperature profile* (energy conservation)

$$\frac{dT}{dr} = -\frac{3}{16}\frac{\kappa(r)\ \rho(r)}{\sigma_{\mathrm{SB}} T(r)^3}\ \frac{l(r)}{4\pi r^2}$$

Solving the Equations: I

given temperature gradient

$$\frac{dT}{dr} = -\frac{3}{16}\frac{\ell_{\mathrm{mfp}}}{\sigma_{\mathrm{SB}}T(r)^3}\ \frac{l(r)}{4\pi r^2} = -\frac{3}{16}\frac{\kappa(r)\ \rho(r)}{\sigma_{\mathrm{SB}}T(r)^3}\ \frac{l(r)}{4\pi r^2} \tag{10}$$

formally can integrate

$$T(r) = T_{\mathrm{c}} - \frac{3}{16}\int_0^r \frac{\kappa(r)\ \rho(r)}{\sigma_{\mathrm{SB}}T(r)^3}\ \frac{l(r)}{4\pi r^2} dr \tag{11}$$

and similarly we can formally integrate dl/dr to find

$$l(r) = 4\pi \int r^2\ \mathcal{L}(\rho, T) = 4\pi \int r^2\ q(r)\ \rho(r) \tag{12}$$

Q: but why is this not so simple? what's the subtlety?

when we integrate

$$\frac{dT}{dr} = -\frac{3}{16} \frac{\kappa(r)\ \rho(r)}{\sigma_{\mathrm{SB}} T(r)^3}\ \frac{l(r)}{4\pi r^2} \tag{13}$$

this requires both nuclear reaction rates in $\ell(r)$,
and on opacities $\kappa(r)$
that themselves depend on temperature!

example of general lesson:

- stellar structure equations inter-related
- must be solved together
- realistic cases require computers
- but simple models still useful for insight and to check for programming bugs!

Here Endeth the Material on the Hour Exam