Astro 404
Lecture 23
Oct. 21, 2019

Announcements:

- **Problem Set 7 due Friday 5pm**
- Exam: grading elves hard at work

Before Exam:
End of main sequence

- *core density and temperature increases on main sequence*
- but at end of main sequence equilibrium lost!
  core contracts until new pressure source emerges

Conclusion:
need to understand matter at high density and pressure

Thermal Photons: Blackbody Radiation

dimensional analysis: kT, h, c form one *length*

$$\ell = \frac{hc}{kT} = \frac{h}{p_T} \tag{1}$$

the *thermal de Broglie length*

from this we estimate **number density**

$$n_\gamma \sim \ell^{-3} \sim \left(\frac{kT}{hc}\right)^3 \tag{2}$$

energy density

$$\varepsilon_\gamma \sim kT\ell^{-3} \sim \frac{(kT)^4}{(h^3c^3)} \tag{3}$$

pressure has dimensions of energy density, so

$$P_\gamma \sim \varepsilon \tag{4}$$

of course we know thermal photons result: blackbody radiation!

Blackbody Radiation: Exact Results

for blackbody photons at T, with $g = 2$ polarizations:

$$
\begin{aligned}
n_\gamma &= g\frac{\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \propto T^3 \\
\varepsilon_\gamma &= g\frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} = a_{\mathrm{SB}}\, T^4 \\
P_\gamma &= \frac{1}{3}\varepsilon_\gamma \propto T^4
\end{aligned}
$$

where $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3 = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.20206\ldots$

and $\hbar = h/2\pi$ is the chic "reduced Planck's constant"

and $a_{\mathrm{SB}} = \pi^2/15\ k^4/\hbar^3 c^3$ is the Stefan-Boltzmann radiation constant

Note: already saw that relativistic gas has $P = \varepsilon/3$

also note: energy flux is roughly $F \sim c\varepsilon \sim T^4$

which is Stefan-Boltzmann result!

Radiation Pressure

long ago in Lecture 2 we saw that
because photons carry momentum, when they interact
they also exert force: **radiation pressure**

in PS1: pressure force due to flux of photons
all moving outward radially

blackbody radiation: randomly directed thermal photons

$$P_{\rm rad} = \frac{1}{3}\varepsilon_{\rm rad} = \frac{1}{3}a_{\rm SB}T^4 \ \propto \ T^4 \tag{5}$$

exists in stars in addition to gas pressure!

$$P_{\rm tot} = P_{\rm gas} + P_{\rm rad} \tag{6}$$

so for ideal gas and radiation

$$P_{\rm tot} = \frac{\rho\, kT}{m_{\rm g}} + \frac{1}{3}a_{\rm SB}T^4 \tag{7}$$

iClicker Poll: Radiation Pressure in Stars

Vote your conscience! *All answers get credit*

in which main seq stars does radiation dominate pressure?

A the lowest mass stars

B the highest mass stars

C intermediate masses, neither lowest nor highest

D radiation pressure never exceeds gas pressure

Star Average Pressure and Mass

total pressure sums gas and radiation

$$P_{\rm tot} = P_{\rm gas} + P_{\rm rad} = \frac{\rho\, kT}{m_{\rm g}} + \frac{1}{3} a_{\rm SB} T^4 \tag{8}$$

and average mass density

$$\langle \rho \rangle = \frac{3M}{4\pi R^3} \sim \frac{M}{R^3} \tag{9}$$

For ideal gas: Virial theorem $\rightarrow$ average interior temperature

$$\langle kT \rangle = \frac{2}{3} \frac{m_{\rm g} U}{M} \sim \frac{G M m_{\rm g}}{R} \tag{10}$$

so ratio of pressures (PS7)

$$\frac{P_{\rm rad}}{P_{\rm gas}} \propto \frac{T^3}{\rho} \sim \frac{M^3/R^3}{M/R^3} = M^2 \tag{11}$$

so *on main sequence:*

- radiation pressure small for low mass stars (including Sun!)
- *radiation pressure dominates in high mass stars!*

Quantum Matter and Density

now consider a gas of non-relativistic matter
allow quantum effects

non-relativistic: must have $v \ll c$
so for thermal particles, typical kinetic energy $mv^2/2 \sim kT \ll mc^2$

for non-relativistic particles of mass m, at temperature T
typical kinetic energy

$$E_{\mathrm{k}} = \frac{p^2}{2m} \sim kT \tag{12}$$

gives typical *thermal momentum* $p_T \sim \sqrt{m\,kT}$

Q: what is thermal de Broglie wavelength here?
Q: estimate of number density n? mass density ρ?

thermal momentum $p_T \sim \sqrt{m\,kT}$
gives **thermal de Broglie wavelength**

$$\lambda_{\rm deB}(T) = \frac{h}{p_T} \sim \left(\frac{h^2}{m\,kT}\right)^{1/2} \tag{13}$$

and so naively expect a number density

$$n_{\rm naive}(T) \sim \lambda_{\rm deB}(T)^{-3} \sim \left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{14}$$

and mass density

$$\rho_{\rm naive}(T) = m\; n_{\rm naive}(T) \sim m\left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{15}$$

for a given species m, this gives a number density $n(T)$ entirely and universally determined by temperature!

∞ *Q: what is strange about this result?*
Hint: what sets $\rho(T)$*? apply to objects in this room?*

naively expect mass density

$$\rho_{\text{naive}}(T) = m\ n_{\text{naive}}(T) \sim m \left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{16}$$

but that can't be right!
density of water in you, a beverage, and the air are all different!

also: for $T = 300$ K this gives
$n_{\text{naive,water}} \sim 10^{27}\ \text{cm}^{-3}$, and $\rho_{\text{naive,water}} \sim 3\times10^{4}\ \text{g/cm}^3$. *Yikes!*

Q: where did we go wrong?

really: we have assumed particle spacing always around $\lambda_{\mathrm{deB}}(T)$
this is "quantum size" of thermal particles

this sets a special density: the **quantum concentration**

$$n_Q = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \sim \frac{1}{\lambda_{\mathrm{deB}}^3} \tag{17}$$

n_Q rises with T since $\lambda_{\mathrm{deB}}(T) = h/p_T \propto T^{-1/2}$

but clearly

- real particle density can be lower or higher!
- n_Q is high compared to everyday matter

Q: why do we expect physically if $n \ll n_Q$? if $n \gtrsim n_Q$?

• if $n \ll n_Q$:
particle spacings larger than thermal de Broglie wavelength
particles are "too far apart" for quantum effects
expectation:
quantum effects small: ordinary ("classical") ideal gas!

• if $n \gtrsim n_Q$:
particle spacings of same order as de Broglie
now expect departures from classical ideal gas
must include quantum effects

namely: combine Pauli exclusion principle
with Heisenberg uncertainty principle

Identical Particles

experiments and theory show: *all particles of each species are completely identical and* **indistinguishable**

example: all electrons are completely identical
as are all photons, neutrons, protons, etc
always have exactly same charge, mass, spin

spoiler: not just result of a high-quality "electron factory"
but really: space filled with "electron field"
whose quantum excitations are electron particles

Pauli: this has profound effects in quantum mechanics for systems of multiple particles

Q: experts–what's the rule? what does it depend on?

Pauli Principle

behavior of identical particles
depends on spin (particles "self" angular momentum)

Bosons: particles with spin $S = 0, 1, 2, \ldots$
example: photon $S = 1$ is a boson
no restriction on number of boson in same quantum state
"bosons are social" – party anmials of the quantum world

Fermions: spin $S = 1/2, 3/2, 5/2, \ldots$ ("half-integer spin")
ex: electrons, protons, neutrons all are $S = 1/2$ Fermions
at most one Fermion per quantum state
"fermions are loners" – they want to be alone!
Pauli exclusion principle

profound implications for the nature of matter!

Uncertainty Principle

Heisenberg: wave-particle duality means

- cannot know position better than $\sim$ de Broglie wavelength position uncertainty $\Delta x \gtrsim \lambda_{\mathrm{deB}} \sim h/p_x$
- cannot know momentum for particle confined to Δx better than x-momentum uncertainty $\Delta p_x \gtrsim h/\Delta x$

can show in general: **uncertainty principle**

$$\Delta x \ \Delta p_x \ \geq \ h \tag{18}$$

so in volume $\Delta V = \Delta x \ \Delta y \ \Delta z$

$$\Delta V \ \Delta^3 p \ \geq \ h^3 \tag{19}$$

with "momentum space" volume $\Delta^3 p = \Delta p_x \ \Delta p_y \ \Delta p_z$

Q: so what is maximum number density for gas of electrons?

Maximum Fermion Density

Pauli exclusion principle means fermions obey

$$\Delta V \ \Delta^3 p \ \geq \ h^3 \tag{20}$$

so for gas of electrons with $S = 1/2$

- 2 possible spin states $(\uparrow, \downarrow)$
  same energy in both: *degenerate* states
- maximum number density n_e set by

$$n_{e,\mathrm{max}} \ \Delta V = 2 \tag{21}$$

which gives

$$n_{e,\mathrm{max}} = \frac{2}{\Delta V} = \sum_p \frac{2}{h^3} \Delta^3 p \tag{22}$$

momentum space volume $4\pi/3 \ p^3$ has $\Delta^3 p = 4\pi p^2 \ dp$
up to some *maximum momentum* ("Fermi momentum") p_0

Pauli-approved maximum electron density
sums (integrates) all possible momenta up to some p_0

$$n_{e,\mathrm{max}} = \frac{2}{\Delta V} = \frac{2}{h^3} 4\pi \int_0^{p_0} p^2 \, dp \tag{23}$$

$$= \frac{8\pi}{3h^3} \, p_0^3 \tag{24}$$

maximum density also called *degenerate number density*

required maximum momentum to have number density n_e:

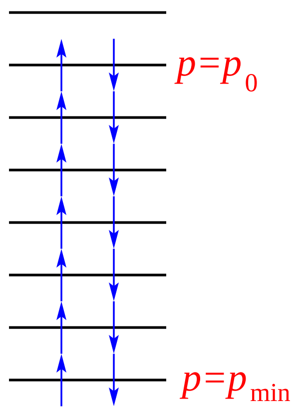
$$p_0 = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \ \sim \ \frac{h}{\ell} \tag{25}$$

so Fermi momentum set by uncertainty principle $p_0 \, \ell \sim h$
where distance $\ell = n_e^{-1/3}$ is *typical particle spacing*

Q: for degenerate gas, what is special about states above, below p_0?

Fermi momentum for electron gas of number density n_e:

$$p_0 = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \;\sim\; \frac{h}{\ell} \qquad (26)$$



number density n_e sets highest momentum
reached by filling all states up to p_0
and leaving all others empty

now consider the case of $p_T = \sqrt{mkT} \gg p_0$

Q: what does this mean physically?

Q: what does this mean for density?

if gas is completely degenerate

$$p_0^3 = \frac{3n_e h^3}{8\pi} \tag{27}$$

so so if $p_0 \ll p_T = \sqrt{mkT}$, then physically
thermally available momentum states far exceed needed p_0
momentum states don't have to be "packed full"
density is not maximal $\rightarrow$ *gas is not degenerate*

quantitatively, we have

$$\frac{3n_e h^3}{8\pi} \ll p_T^3 = (mkT)^{3/2} \tag{28}$$

$$n_e \ll \frac{8\pi}{3}\left(\frac{mkT}{4\pi^2\hbar^2}\right)^{3/2} \sim n_Q \tag{29}$$

lesson: *non-degenerate* $\Leftrightarrow$ density $\ll$ quantum concentration

so air in this room, gas in solar core today: non-degenerate