Astro 404
Lecture 24
Oct. 23, 2019

Announcements:

- **Problem Set 7 due Friday 5pm**
- Office Hours: instructor today 11am-noon or by appointment
- TA: Thursday noon-1pm
- Exam: grading elves hard at work

Last time: final piece of stellar physics puzzle

matter and radiation at high density and pressure

- matter: classical and quantum gasses
  dividing line: quantum concentration n_Q
  Q: what's that? matter behavior at $n < n_Q$*?* $n > n_Q$*?*

quantum concentration:
particle number density when particle spacing is thermal de Broglie wavelength

$$\lambda_{\mathrm{deB},T} = \frac{h}{p_T} = \sqrt{\frac{h^2}{m\,kT}} \tag{1}$$

which gives a number density:

$$n_Q = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \sim \lambda_{\mathrm{deB},T}^{-3} \tag{2}$$

expect: quantum behavior when $n > n_Q$
non-quantum (classical) behavior when $n < n_Q$

Degeneracy Revisited: Building Quantum Systems

central result of quantum mechanics:
when quantum particles confined to finite volume of space
not all energies are allowed!

allowed states have definite energies: "energy levels"
which may or may not be different for different spin states

Pauli Principle: at most one Fermion per quantum state
including both energy and spin
if energy levels the same for spin up and down
then two particles per energy level

ω example: atoms
Q: what confines atom particles to small volume?

Filling Energy Levels: Electrons in Atoms

atoms made of *nucleus* of charge $Q = +Ze$
surrounded by Z *electrons*, each of charge $Q = -e$
with $Z = 1$ for hydrogen, $Z = 92$ for uranium

electrostatic (Coulomb) attraction binds electrons to nucleus
and thus confines them

hence:

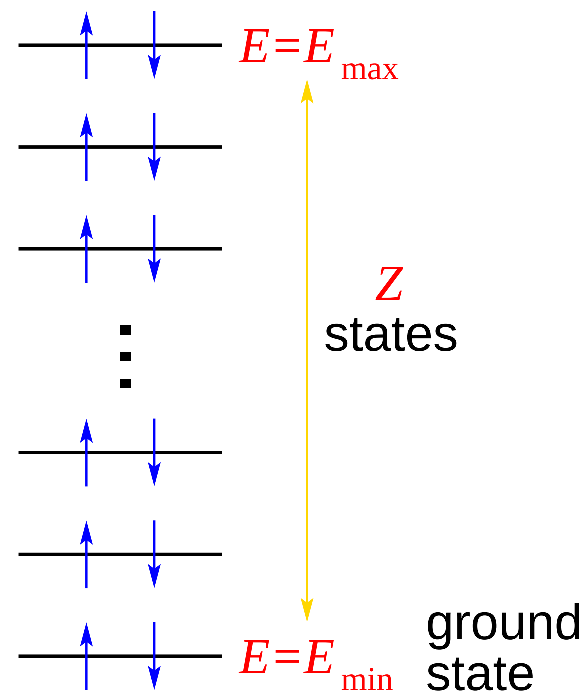
- electrons in atoms have discrete energy levels
  that is, energy levels not continuous, come in "steps"
  and thus only some electron "orbits" allowed
- electrons are Fermions: only one per quantum state
- energy roughly independent of spin: 2 state per energy level

Q: how to "build" a normal (unexcited) atom?

Building an Atom

to "build" an atom:

- lowest energy level: **ground state**
  fits up to 2 electrons, spins ↑↓
  these have same energy: "degenerate"
- for normal (unexcited) atom:
  keep adding electrons
  two per energy level
  from the lowest available energy up
- after ground state, fill first excited state
- repeat until all Z electrons added



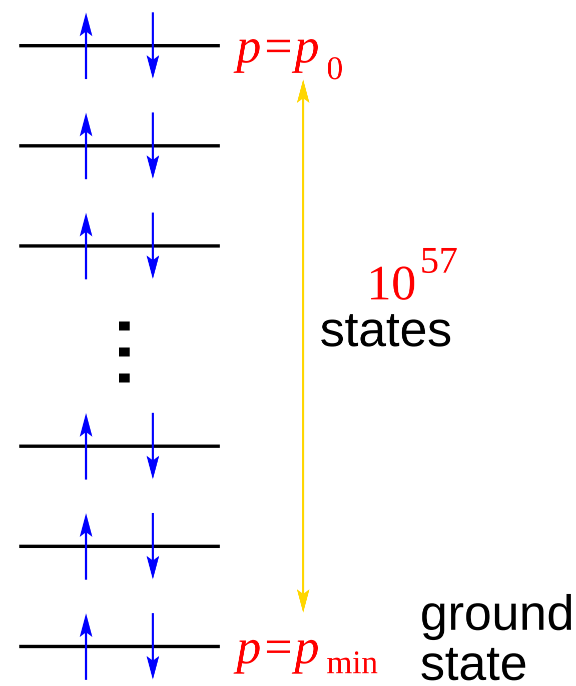
compare and contrast to degenerate star with $\sim 10^{57}$ atoms

Q: what confines the electrons? what sets levels? how to fill?

A Degenerate Electron Gas

consider a degenerate star: a $N \approx 10^{57}$ particle gas
of free particles in volume confined by gravity!

"cold degenerate gas" $T = 0$



free quantum particles: states labelled by
de Broglie wavelength $\lambda = h/p$
or equally well by *momentum* p or energy $E(p)$

to build star out of degenerate electron gas:

- start in ground state
  adding 2 electrons $\uparrow\downarrow$
- fill first excited state, etc

- until reach highest level: Fermi level
  labelled by Fermi momentum p_0 or Fermi energy $E(p_0)$

Counting Degenerate States: Number Density

last time: Heisenberg says number of free electron states in volume V is: $\mathcal{N}_e = n_e V$

where electron number density sums up states from ground state to Fermi level:

$$n_e = n_{e\,\mathrm{degen}} = \frac{2}{h^3} \int_0^{p_0} d^3p = \frac{8\pi}{h^3} \int_0^{p_0} p^2 \; dp \qquad (3)$$

$$= \frac{8\pi}{3h^3} \, p_0^3 \qquad (4)$$

particle states filled with maximum efficiency up to p_0

$\Rightarrow$ highest density possible for this number of electrons

- so Fermi level sets maximum number density
  but also works the other way

to create a degenerate gas of electrons with number density n_e
requires Fermi momentum

$$p_0 = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \;\sim\; \frac{h}{\ell} \tag{5}$$

where $\ell = 1/n_e^{1/3}$ is the typical electron spacing

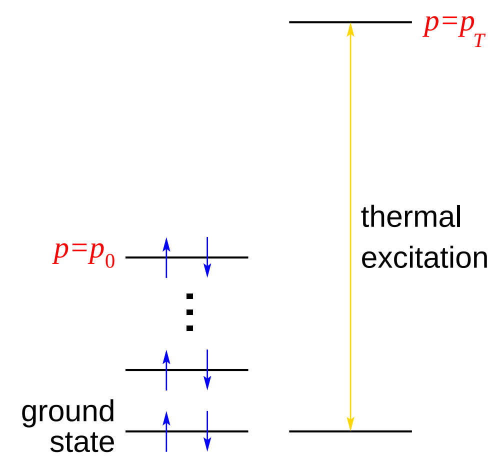
number density n_e sets highest momentum
reached by filling all states up to p_0
and leaving all others empty

now consider T nonzero, with $p_T = \sqrt{mkT} \gg p_0$
Q: what does this mean physically?
Q: what does this mean for density?

if gas is completely degenerate

$$p_0^3 = \frac{3 n_e h^3}{8\pi}$$

so if $p_0 \ll p_T = \sqrt{mkT}$, then physically
thermal excitations of momentum states
far exceed needed p_0
momentum states don't have to be "packed full"
and uncertainty principle allows larger spacing
density is not maximal $\rightarrow$ *gas is not degenerate*



so *heating a degenerate gas* to $kT \gg E(p_0)$
"lifts" the degeneracy $\rightarrow$ recover classical ideal gas
this will be explosively crucial for the fate of Sun-like stars!

At What Density is Onset of Degeneracy?

quantitatively, to lift degeneracy with $p_T \gg p_0$ we have

$$p_0^3 = \frac{3 n_e h^3}{8\pi} \ll p_T^3 = (mkT)^{3/2} \quad (6)$$

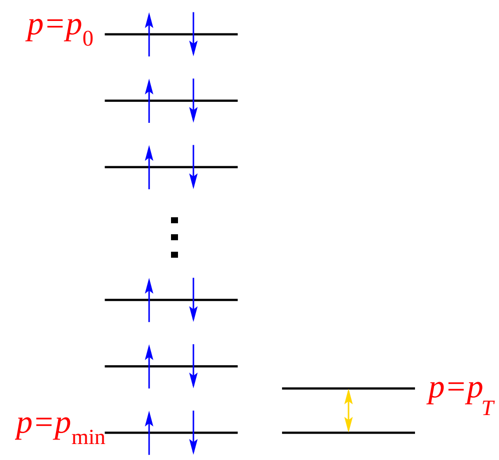
$$n_e \ll \frac{8\pi}{3} \left(\frac{mkT}{4\pi^2 \hbar^2}\right)^{3/2} \sim n_Q \quad (7)$$

lesson: *non-degenerate* $\Leftrightarrow$ density $\ll$ quantum concentration

so air in this room, gas in solar core today: non-degenerate

Q: what if $p_T = \sqrt{mkT} \ll p_0$*?*

a *cold gas* has $p_T \ll p_0$



thermal excitations far below Fermi energy
still need to "tightly pack" energy levels
gas is degenerate

this agrees with our derivation of degenerate state filling that implicitly used $T = 0$

density condition: degenerate if $n > n_Q$

iClicker Poll: Nuclei/Ions

Vote your conscience! *All answers get credit*

So far we have focussed on degenerate electron gas
What about the nuclei they came with?

What sets the reationship between nuclei and e?

A must balance energy: energy densities must be equal

B must balance momentum: Fermi momenta must be equal

C must balance pressure: pressures must be equal

D must balance charge: p and e numbers must be equal

Mass Density of a Degenerate Electron Gas

electron number density: $n_e = 8\pi/3h^3 \; p_0^3$

electron mass density

$$\rho_e = m_e n_e = \frac{8\pi}{3}\frac{m_e p_0^3}{h^3} \tag{8}$$

but there are nuclei (positive ions), giving net charge zero!
so total p and e densities must balance: $n_{p,\mathrm{tot}} = n_e$

if average ion charge is Z_i and mass $m_i = A_i m_p$:
total proton number density $n_p = Z_i n_i$
ion number density $n_i = n_e/Z_i$
ion mass density $\rho_i = m_i n_i = A_i m_p n_e / Z_i$

total mass density $\rho = \rho_e + \rho_i \approx \rho_i$: dominated by ions

Pressure of a Degenerate Electron Gas

in studying ideal gas, found that pressure
is an average momentum flow:

$$P = \text{momentum per particle} \times \text{particle flux} = \frac{1}{3}\langle p \ v \ n\rangle \tag{9}$$

where $v(p)$ is the velocity for momentum p
if *non-relativistic:* $v = p/m$

for degenerate electron gas, pressure is

$$P_e = \frac{1}{3}\int p \ v \ dn_e = \frac{8\pi}{3h^3}\int_0^{p_0} p \ v \ p^2 \ dp \tag{10}$$

$$= \frac{8\pi}{15 m_e h^3} \, p_0^5 \tag{11}$$

but Fermi momentum given by number density: $p_0 \sim n_e^{1/3} h$

$$P_e = \frac{8\pi h^2}{15 m_e}\left(\frac{3 n_e}{8\pi}\right)^{5/3} \tag{12}$$

Non-Relativistic Degeneracy Pressure

for (cold) non-relativistic degenerate electrons

$$P_{e,\mathrm{nr}} = \frac{8\pi h^2}{15 m_e} \left(\frac{3 n_e}{8\pi} \right)^{5/3} \tag{13}$$

- pressure only depends on density and not temperature
- pressure grows with density

  $P_{e,\mathrm{nr}} \propto n_e^{5/3} \propto \rho^{5/3}$
- degeneracy pressure is large even when temperature small!
  due to Pauli principle! a quantum effect!
  contrast classical ideal gas: $P = nkT \to 0$ as $T \to 0$
- sometimes useful to write $P_e = K_{\mathrm{nr}}\, n_e^{5/3}$, with

$$K_{\mathrm{nr}} = \left(\frac{3}{8\pi} \right)^{2/3} \frac{h^2}{5 m_e} \tag{14}$$

Q: what changes if the electrons are relativistic?

iClicker Poll: Relativistic Pressure

Vote your conscience! *All answers get credit*

Non-relativistic degenerate gasses have $P_{\text{nr}} \propto \rho^{5/3}$

What will we find for relativistic degenerate gasses?

A also $P_{\text{rel}} \propto \rho^{5/3}$

B P_{rel} increases more strongly with ρ

C P_{rel} increases more less with ρ

Now Go Relativistic

if electrons are relativistic: speeds $v \approx c$, momenta $p \gg m_e c$
and energy $E = \sqrt{(m_e c^2)^2 + (cp)^2} \to cp$

but free particle states still set by de Broglie wavelength
and still labelled by momentum $\lambda_{\rm deB} = h/p$

number density still $n_e = 8\pi/3 \; p_0^3/h^3$
pressure uses $v(p) \approx c$:

$$P_{e,\rm rel} = \frac{1}{3}\langle p \; v \; n\rangle = \frac{2\pi}{3}\frac{cp_0^4}{h^3} \quad (15)$$

$$= \frac{2\pi}{3} hc \left(\frac{3n_e}{8\pi}\right)^{4/3} \quad (16)$$

- relativistic pressure also only depends on density and not T
- pressure grows with density $P_{e,\rm rel} \propto n_e^{4/3} \propto \rho^{4/3}$
- weaker power law than non-relativistic scaling
- $P_{\rm rel} = K_{\rm rel} \; n_e^{4/3}$ with $K_{\rm rel} = (3/8\pi)^{1/3} hc/4$

Polytropes

We see that cold degenerate gasses have pressure
- that depends only on density
- and varies as a power law: $P \propto \rho^{\gamma}$

generalize to *polytropic equation of state*

$$P = K\ \rho^{\gamma} = K\ \rho^{1+1/n} \tag{17}$$

- pressure depends only on density, with
- K a constant
- n called the polytrope index

simplifies stellar structure!

Hydrostatic Equilibrium of a Polytrope

use polytropic equation of state

$$P = P(\rho) = K\ \rho^{\gamma} = K\ \rho^{1+1/n} \tag{18}$$

hydrostatic equilibrium gives

$$\frac{dP}{dr} = -\frac{Gm(r)\ \rho(r)}{r^2} \tag{19}$$

Q: what determines $m(r)$*?*

Q: so what is special about a polytrope?

hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{Gm(r)\ \rho(r)}{r^2} \tag{20}$$

- enclosed mass $m(r) = 4\pi \int r^2\ \rho(r)\ dr$ depends on ρ
- so righthand side depends only on ρ and r
- for polytrope $P(\rho)$ depends only on ρ
- so lefthand side also depends only on $\rho(r)$

lesson: this equation ultimately depends only on ρ and r
solving it gives $\rho(r)$: determines star density structure!

note that this won't work if $P = P(\rho, T)$
then need to find temperature $T(r)$ too
this is what's special about a polytrope