Astro 404
Lecture 25
Oct. 25, 2019

Announcements:

- **Problem Set 7 due today 5pm**
- last minute instructor Office Hours after class
- **Problem Set 8 due Fri Nov 1**

Last time: Congratulations!
last piece of physics in place!
recall: stars interweave all four fundamental forces
and require understanding matter at high and low ρ, P, T
and at scales from subatomic to astronomical
and you now have the know-how! ...but based on HW questions,
a few more words might help.

What Makes a Particle Relativistic?

individually and in groups
particles behave differently if they are relativistic vs not but how do we know when these apply?

Newtonian physics: developed for slow particles: $v \ll c$
these are non-relativistic, and familiar

for particle of mass m

- non-relativistic momentum $p_{\mathrm{nr}} = mv$
- non-relativistic kinetic energy $E_{\mathrm{k,nr}} = mv^2/2 = p^2/2m$

Special Relativity

Einstein: Newtonian physics fails when speeds $v \to c$
need to rethink particle dynamics!

- **relativistic momentum**

$$p_{\text{rel}} = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

- **relativistic total energy**

$$E_{\text{rel}} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \sqrt{(cp_{\text{rel}})^2 + (mc^2)^2}$$

Q: Einstein results for $v = 0$? for $v \ll c$?

relativistic momentum $p_{\text{rel}} = mv/\sqrt{1 - v^2/c^2}$

relativistic energy $E_{\text{rel}} = mc^2/\sqrt{1 - v^2/c^2}$

for $v = 0$:

$p_{\text{rel}} = 0$: agrees with Newtonian result for $v = 0$

$E_{\text{rel}} = mc^2$: even with no motion, particle mass contains energy!

"**rest mass energy**" or "rest energy" is $E_{\text{rest}} = mc^2$

for $v \ll c$:

$p_{\text{rel}} \to mv$ – recover Newtonian result!

use Maclaurin expansion $(1 - v^2/c^2)^{-1/2} = 1 + 1/2\ v^2/c^2 + \cdots$

$$E_{\text{rel}} = mc^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2 + \cdots = E_{\text{rest}} + E_{\text{k,nr}}$$

energy just due to motion is Newtonian kinetic energy!

iClicker Poll: When are We Relativistic?

Which of these indicates a particle is relativistic?

that is, need to use special relativity momentum, energy?

A $v \sim c$

B $p \gtrsim mc$

C $E_\mathrm{k} \gtrsim mc^2$

D (a), (b), and (c) are all true if any are true

Newtonian limit good for $v \ll c$
will fail when v becomes close to c

example: for what v is *relativistic momentum* $p = mc$?

$$p_{\text{rel}} = \frac{mv}{\sqrt{1 - v^2/c^2}} = mc \tag{1}$$

when $v = c/\sqrt{2} = 0.707c$: 70.5% of lightspeed
and $E = \sqrt{2}mc^2$, so that $E_{\text{k,rel}} = 0.414\ mc^2$

confirms: relativistic effects large when $v \sim c$
and shows this is equivalent to: $p_{\text{rel}} \gtrsim mc$, $E_{\text{k,rel}} \gtrsim mc^2$
PS7: use p/mc as test of how relativistic

Q: what about ultra-relativistic limit $v \to c$?
Q: what about massless particles?

ultra-relativistic limit: $v \to c$:

- $p \to mc/\sqrt{1 - v^2/c^2} = E/c$
- $E \gg mc^2$

for photons (massless particles):

use $E = \sqrt{(cp)^2 + (mc^2)^2} = cp$

same as ultrarelativistic limit for massive particle!

Q: whats the connection between non-relativistic vs relativistic and degenerate vs non-degenerate gasses?

for gasses:

- non-relativistic: $\langle v \rangle \ll c$
- relativistic: $\langle v \rangle \sim c$ (or $v = c$ if photons)

these conditions are *independent of*:

- non-degenerate: particle spacing $\gg$ "quantum particle reach"
  that is: $\ell \sim n^{-1/3} \gg \lambda_{\rm deB} = h/p$ de Broglie wavelength
  that is: $n < n_Q \sim (mkT/h^2)^{3/2}$ quantum concentration
- degenerate: $n > n_Q$ "maximal density"

so: four possibilities

- non-relativistic and non-degenerate
- relativistic and non-degenerate
- non-relativistic and degenerate
- relativistic and degenerate

∞

stars sample all of these cases!

Polytropes

We see that cold degenerate gas pressure

- that depends only on density
- and varies as a power law: $P \propto \rho^\gamma$

generalize to *polytropic equation of state*

$$P = K\ \rho^\gamma = K\ \rho^{1+1/n} \tag{2}$$

- pressure depends only on density, with
- K a constant
- $n = 1/(\gamma - 1)$ called the polytrope index

no temperature dependence: simplifies modeling
mechanical structure decoupled from thermal structure

Hydrostatic Equilibrium of a Polytrope

use polytropic equation of state

$$P = P(\rho) = K\ \rho^{\gamma} = K\ \rho^{1+1/n} \tag{3}$$

hydrostatic equilibrium gives

$$\frac{dP}{dr} = -\frac{Gm(r)\ \rho(r)}{r^2} \tag{4}$$

isolate enclosed mass term

$$\frac{r^2}{\rho}\frac{dP}{dr} = -Gm(r) \tag{5}$$

now recall enclosed mass $dm = 4\pi r^2\ \rho\ dr$ so

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho \tag{6}$$

hydrostatic equilibrium, rewritten

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho \tag{7}$$

for polytrope $P = K\rho^\gamma$, can write this as

$$\frac{K}{4\pi G}\ \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho^\gamma}{dr}\right) = -\rho \tag{8}$$

a version of the **Lane-Emden equation**

a differential equation that only depends on ρ

- so solution gives $\rho(r)$
- which also gives $P(r) = K\ \rho(r)^\gamma$

$\Rightarrow$ determines structure of star!

PS8: solve Lane-Emden for a simple-ish case

A Non-Relativistic Degenerate Star

consider a star of mass M and radius R
made of a non-relativistic degenerate gas
so pressure is $P = K_{\rm nr}\ n_e^{5/3}$

equate this to the central pressure $P_c \sim GM^2/R^4$:

$$K_{\rm nr} \left(\frac{M}{R^3}\right)^{5/3} \sim \frac{GM^2}{R^4} \tag{9}$$

$$K_{\rm nr} \frac{M^{5/3}}{R^5} \sim \frac{GM^2}{R^4} \tag{10}$$

so the stellar radius:

$$R \sim \frac{K_{\rm nr}}{G}\ M^{-1/3} \tag{11}$$

and for $M = 1 M_\odot$ with 2 nucleons per electron, estimate

$$R_{\rm degen}(1M_\odot) \sim\ 10^4\ {\rm km} \sim 2\,{\rm R}_{\rm Earth} \tag{12}$$

Q: what does this imply? are there objects like this?

White Dwarfs: Degenerate Stars

we see that a degenerate star is *incredibly compact!*

$$R_{\text{degen,nr}} \sim \frac{K_{\text{nr,degen}}}{G} \frac{1}{M^{1/3}} \quad (13)$$

$$R_{\text{degen,nr}}(1M_\odot) \sim 10^4 \text{km} \sim 2\text{R}_{\text{Earth}} \quad (14)$$

mass of the Sun packed into Earth-sized volume
as expected for a maximally dense object

compared to stars we know, radius is:

- *tiny* compared to the Sun, giants, and supergiants
- but is exactly in line with white dwarfs

white dwarfs are degenerate stars!

- supported by degeneracy pressure
- somehow resulting from incredible compression which left high temperature (hence white)

White Dwarfs Observed

nearest white dwarf is **Sirius B**: unseen by naked eye
but companion of Sirius A, brightest star in sky

binary system, so mass known: $M(\text{Sirius B}) = 1.02 M_{\odot}$
but radius about Earth-sized! (PS7)

`www: Sirius B in optical`
`www: Sirius B in X-ray` – outshines Sirius A!
Q: what does this mean?

mass-radius relation:

$$R_{\text{degen,nr}} \sim \frac{K_{\text{nr,degen}}}{G} \frac{1}{M^{1/3}} \tag{15}$$

Q: radius if more massive? less? how to test?

White Dwarfs Radius and Mass

white dwarfs: $R_{\rm nr,degen} \sim M^{-1/3}$
so larger mass means smaller radius!
white dwarfs get more compact when adding mas!

to test: compare radii for white dwarfs with different masses

40 Eridani B: Trekkers–this is Vulcan's system, with a confirmed planet!

$$M(\text{40 Eri B}) = 0.50 M_\odot \approx M(\text{Sirius B})/2 \quad (16)$$
$$R(\text{40 Eri B}) = 1.7 R(\text{Sirius B}) \quad (17)$$

indeed smaller mass → smaller radius

Q: how does average density depend on mass for a white dwarf?
Q: what if we keep adding mass to a white dwarf?

White Dwarfs: Increasing Mass

white dwarfs: $R_{\rm nr,degen} \sim M^{-1/3}$
so average density grows with mass!

$$\rho_{\rm nr,degen} \sim \frac{M}{R^3} \propto M^2 \tag{18}$$

adding mass $\rightarrow$ smaller size, higher density
eventually density so high: Fermi level $p_0 \sim n_e^{1/3} h \gg m_e c$
star becomes relativistic degenerate!

for relativistic degenerate gas: $P_{\rm rel,degen} = K_{\rm rel,degen}\ \rho^{4/3}$
and equating this to $P_c \sim GM^2/R^4$ gives

$$K_{\rm rel,degen} \left(\frac{M}{R^3}\right)^{4/3} \sim \frac{GM^2}{R^4} \tag{19}$$

$$K_{\rm rel,degen} \frac{M^{4/3}}{R^4} \sim \frac{GM^2}{R^4} \tag{20}$$

$$M^{2/3} \sim \frac{G}{K_{\rm rel,degen}} \tag{21}$$

hydrostatic equilibrium in degenerate relativistic star gives mass

$$M \sim \left(\frac{G}{K_{\text{rel,degen}}}\right)^{3/2} \tag{22}$$

- radius drops out!
- also independent of density
- only a single, unique mass works!

numerically:

$$M = M_{\text{Chandra}} = 1.4 M_\odot$$

Chandrasekhar limit! S. Chandrasekhar 1931(!!)

Q: what if white dwarf has $M < M_{\text{Chandra}}$?

Q: what if white dwarf has $M > M_{\text{Chandra}}$?

if high-density WD has $M < M_{\text{Chandra}}$
then pressure (more than) enough to balance gravity
$\rightarrow$ *WD is stable against collapse*

but: *if high-density WD has* $M > M_{\text{Chandra}}$
then pressure *not enough* to balance gravity
$\rightarrow$ gravity force not balanced
$\rightarrow$ *star unstable* $\rightarrow$ *collapses under its own weight!*
$\rightarrow$ catastrophe!

conclusion: *Chandrasekhar mass is*
maximum *mass of white dwarfs!*
i.e., most massive possible "ordinary solid" = supported by e degeneracy

Confirmed! All observed white dwarfs have $M < M_{\text{Chandra}}$

White Dwarf Cooling

White dwarfs are hot–hence the name
but supported by degeneracy pressure
not by heating from nuclear reactions

no energy source $\rightarrow$ white dwarfs cool over time
$\Rightarrow$ a white dwarf *is* like a cup of coffee!

luminosity gives energy loss: $L = 4\pi R^2 \sigma T_{\rm eff}^4$
but for non-relativistic white dwarfs $R \propto M^{-1/3}$, so

$$L_{\rm wd} \propto M^{-2/3} T_{\rm eff}^4$$

- at fixed $T_{\rm eff}$ more massive $\rightarrow$ *less luminous!*
- degeneracy pressure nearly independent of temperature
  so cooling doesn't change R, only $T_{\rm eff}$
  Q: so for a fixed WD mass, how does $L_{\rm wd}$ change?

White dwarf luminosity:

$$L_{\rm wd} \propto M^{-2/3} T_{\rm eff}^4$$

for fixed mass cooling drops $T_{\rm eff}$
and thus drops $L_{\rm wd} \propto T_{\rm eff}^4$

so on a plot of $y = \log L$ vs $x = \log T_{\rm eff}$
we expect white dwarfs of a fixed mass to fall on line:
"white dwarf cooling sequence"

$$y = \log L = 4 \log T_{\rm eff} - \frac{2}{3} \log M + {\rm const} \quad (23)$$

$$= 4x - \frac{2}{3} \log M + {\rm const} \quad (24)$$

so test by looking at HR diagram

Q: expectations if all WD have same mass? if a range of masses?

`www: Gaia white dwarfs`