Astronomy 596/496 PC   Spring 2010
Problem Set #2

Due in class: Friday, Feb. 19

Total points: 12+1

ASTR496PC Students: You may drop choose to drop all of any numbered problem (i.e., all of its sub-parts if it has them). But it is recommended that you try them all.

1. *Olber's Paradox.* Prior to Hubble's enlarging of the cosmic distance scale by discovering spiral nebulae are "island universes," it was implicitly assumed that the universe was static, infinitely large, infinitely old, and filled with (unchangingly luminous) stars; let's call this the "naïve cosmology." However, J. de Cheseaux in 1744, and more famously Heinrich Olbers in 1826, noticed that this seemingly straightforward extrapolation of the observed celestial sphere leads to predictions so grossly incorrect that any naked-eye glance the night sky can rule them out.

   We wish to find the brightness of the night sky in the naïve cosmology. The total, wavelength-integrated surface brightness (intensity, or flux per unit angular area Ω) is $I = dF/d\Omega = dE/dAdtd\Omega$. Radiation transfer tells us that as light propagates along some sightline path s, the intensity changes as

$$\frac{dI}{ds} = -n_{\rm abs}\sigma_{\rm abs}I + q \tag{1}$$

   where any sources have luminosity density per unit solid angle $q = dE/dV\,dtd\Omega$, and any absorbing medium has an absorber number density $n_{\rm abs}$ and the cross section $\sigma_{\rm abs}$ of a single absorber, so that the absorption mean free path $\ell_{\rm mpf} = 1/n_{\rm abs}\sigma_{\rm abs}$. Note that in this naïve universe (but not in ours!) we ignore expansion, redshifting, and time dilation.

   (a) **[1 point]** To get a feel for eq. (1), consider various limits.

   (i) First, if light propagates in a region where there are no sources and no absorbers, then $q = 0 = n_{\rm abs}$. This gives $I(s)$ constant along the line of sight: the "conservation of surface brightness." Explain this physically, and why surface brightness does not obey the inverse square law.

   (ii) Now consider a source-free sightline of length d with a constant density of absorbers. Find the resulting $I(d)$ in terms of the initial intensity $I(0)$, and interpret your result physically.

   (iii) Now consider a sightline of length d in which there are no absorbers, and no initial luminosity $I(0) = 0 = n_{\rm abs}$, but there is a uniform distribution of sources q. Find the resulting $I(d)$ and interpret your result physically.

   (iv) Finally, consider a sightline of length d in which there is no initial luminosity $I(0) = 0$, but there is a uniform distribution of sources q and of absorbers $n_{\rm abs}$. Find the resulting $I(d)$ and interpret your result physically.

   (b) **[1 point]** In the naïve cosmology, consider a case in which there is a uniform distribution of stars like the sun, so $q = L_\odot n_\star/4\pi$, with $n_\star$ the number density of stars. Assume $n_{\rm abs} = 0$. Use the results from part (a) to find an expression for the (uniform) sky brightness $I(d \to \infty)$ in such a universe. Interpret your result; what is the physical reason for your very unphysical answer?

(c) **[1 point]** Your answer for part (b) is too simple even for the naïve cosmology, since even in the absence of interstellar matter, the stars themselves can absorb light. Thus $n_{\rm abs} = n_\star$, and the cross section σ is the geometric cross section of a star with radius $R = R_\odot$. Use the results from part (a) to find the $I(d \to \infty)$ in such a universe. You should find your answer is independent of $n_\star$.

(d) **[1 point]** To interpret your result from part (c), it is useful to compare with the surface brightness $I_\odot$ of the sun. To find this, note that the material at the sun's surface emits isotropically and thus equally in every patch of solid angle $d\Omega = \sin\theta d\theta d\phi$. Note also the solar flux $F_\odot$ is the component of the emission that is in the outward radial direction; taking this as the $\hat{z}$ direction at the point of emission, this means that $F_\odot = I_\odot \int_{\theta>0} \cos\theta \, d\Omega$, where the outward directions have $\theta \in [0, \pi/2]$. Use this to find $I_\odot$ in terms of $F_\odot$ and then in terms of $L_\odot$ and $R_\odot$. Finally, express your answer to part (c) in terms of $I_\odot$.

Interpret your result physically. What physically leads to Olber's paradox in the naïve universe? What effect(s) solve the paradox in a big-bang universe? Comment on the cosmological information encoded in the seemingly simple fact that the night sky is dark.

2. **[1 point]** *Newtonian Escape.* In our usual Newtonian cosmology, find the escape speed $v_{\rm esc}$ for a test particle at an arbitrary distance R from some arbitrary cosmic point, enclosing a mass $M(R)$. Find the ratio of the escape speed to the Hubble speed v_H; your result should be independent of R. Interpret your result physically, and be sure to discuss the three cases $v_{\rm esc}/v_H > 1$, < 1, and $= 1$.

3. *The Robertson-Walker Metric.* Different people adopt different conventions for writing down the RW metric, not to mention different coordinate systems. This exercise is to make you familiar with how to shift among them.

(a) **[1 point]** The form I mostly use is more or less that of Peebles,

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2/R^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \tag{2}$$

Peacock and Kolb & Turner write this form as

$$ds^2 = dt^2 - \mathcal{R}(t)^2 \left(\frac{du^2}{1 - \kappa u^2} + u^2 d\theta^2 + u^2 \sin^2\theta d\phi^2 \right) \tag{3}$$

Note that throughout, I am freely using units where $c = 1$.

Show how to go between these metrics: What is the relationship between a and $\mathcal{R}$; what are their units? What is the relationship between r and u; what are their units? For the three values of κ, what are the range of possible values of r? of u?

(b) **[1 point]** Peacock's preferred form for RW is

$$ds^2 = dt^2 - \mathcal{R}(t)^2 \left[d\chi^2 + S_\kappa(\chi)^2 d\theta^2 + S_\kappa(\chi)^2 \sin^2\theta d\phi^2 \right] \tag{4}$$

show how to go between χ and r, and verify the functional forms

$$\begin{aligned} S_{-1}(\chi) &= \sinh\chi & (5)\\ S_0(\chi) &= \chi & (6)\\ S_{+1}(\chi) &= \sin\chi & (7) \end{aligned}$$

Over what values does χ run for each case?

(c) **[1 point]** Note that for $\kappa = +1$, $\chi \in (0, \pi)$. For the case of a 2-D sphere embedded in 3-D, our spatial coordinates become θ and r or u or χ. In this case, draw a sketch showing the r, θ, and χ coordinates. Use the sketch to illustrate the physical significance of the regions with $\chi = 0, \pi/2, \pi$. Explain the problem/subtlety with the r (or u) coordinate in case of $\kappa = +1$.

Go on to calculate the comoving spatial volume of the universe for $\kappa = +1$, and show it to be $V_3 = 2\pi^2 R^3$.

Finally, what is the comoving spatial volume for universes with $\kappa = 0$ or $\kappa = -1$?

4. *Horizons.* As discussed in class, particle horizons play a key role in cosmology.

(a) **[1 point]** Show that the comoving distance d_{horiz} traversed by a photon from the beginning $(t = 0, r = r_1)$ until $(t_0, r = 0)$ is given by

$$d_{\text{horiz}} = \int_0^r \sqrt{g_{rr}}\,dr = \int_0^{t_0} \frac{dt}{a(t)} = \eta(t_0) \qquad (8)$$

Briefly explain why it is sensible to use this distance to define the *particle horizon.*

(b) **[1 point]** Find expressions for $d_{\text{horiz}}(t)$ in a matter-dominated and in radiation-dominated universe. In each case, what is the behavior of d_{horiz} as $t \to 0$? Interpret this result physically, and suggest why one might naïvely have expected the opposite result. Comment on the relevance to the cosmic microwave background.

(c) **[1 point]** In addition to a particle horizon, it is sometimes useful to define a cosmic *event* horizon, via

$$d_{\text{event}} = \int_{t_0}^{\infty} \frac{dt}{a(t)} \qquad (9)$$

Interpret the physical significance of d_{event}. For matter- and radiation-dominated universe, find d_{event} in the limit $t \to \infty$, and comment.

Finally, for a universe (like ours) that is Λ-dominated, find d_{event} in the limit $t \to \infty$, and comment on the (far) future of observational cosmology.

(d) **[1 bonus point]** For a closed universe containing only matter, show that a photon born in the big bang can circumnavigate the universe and arrive back at its starting point just at the big crunch.

5. *Conformal Time.* **[1 point]** Just as different spatial variables are useful in different circumstances, it is sometimes useful to introduce a new time variable, the conformal time η defined by $d\eta = dt/a(t)$. Find $a(\eta)$ and $\eta(a)$ for the cases of universes dominated by matter, by radiation, and by Λ.