Astronomy 596 PC   Spring 2010
Problem Set #3
Due in class: Friday, March 5
Total points: 10+2

Dark Energy and the Cosmological Constant. This problem set will explore various issues surrounding the evidence for an accelerating universe, and its implications for dark energy.

1. *The Einstein Static Universe* As mentioned in class, Einstein initially invented the Cosmological constant in order to "fix" the dynamic nature of a homogeneous and isotropic spacetime and to keep the universe static and non-expanding.

   (a) **[1 point]** Consider the Friedmann equations with non-relativistic matter, a (possibly) nonzero curvature term, and a cosmological constant. Given the density ρ_0 of the universe (now constant in space *and* time), and assuming $P = 0$, find the critical value Λ_c of the cosmological constant which makes the universe static, so that $\dot{a} = \ddot{a} = 0$. Since the scale factor is constant, it keeps its present value $a_0 = 1$ always. This solution is called the *Einstein static universe.* Also show that in this model, we must have $k = +1$, i.e. the universe is not only static but finite in size. Give an expression for the curvature radius R_0 in terms of Λ_c.

   (b) **[1 bonus point]** Now consider an Einstein static universe in which there is a small perturbation to the scale factor, $a = 1 + \epsilon$, with $\epsilon \ll 1$. Expand the Friedmann acceleration equation to first order in ϵ. Show that the solution has ϵ growing exponentially with time, and find the growth timescale.
   What does this result imply for the static universe?

2. *The Cosmological Constant and Everyday Life.* With Λ, the laws of gravity change, even in the Newtonian limit. Poisson's equation becomes

$$-\nabla \cdot \vec{g} = \nabla^2 \phi = 4\pi G\rho - \Lambda c^2 \tag{1}$$

   (a) **[1 point]** Assume spherical symmetry, so that $\phi = \phi(r)$ and $\vec{g} = g(r)\hat{r}$. Show that in empty space ($\rho = 0$), the gravitational acceleration $\vec{g}_\Lambda = -\nabla\phi_\Lambda$ on a particle at $\vec{r}$ is nonzero, *repulsive*, and linear proportional to r!
   Explain why a repulsion a physically reasonable result. Since there is no matter, this repulsion is attributed to the vacuum itself, and is what is meant in popular press discussion of cosmic "antigravity."

   (b) **[1 point]** Since eq. (1) does not agree with the Newtonian limit, we might expect trouble even within the dynamics of the solar system. To investigate this, consider the orbit of the Earth around the Sun. Find the ratio $g_\Lambda / g_{\rm Newt}$ of the magnitudes of the cosmological constant's repulsive acceleration g_Λ the usual attractive acceleration $g_{\rm Newt}$. You may take the Earth's orbit to be circular, with radius r. Show that the ratio can be written

$$\frac{g_\Lambda}{g_{\rm Newt}} = \Omega_\Lambda \left(\frac{H_0 r}{v_c}\right)^2 \tag{2}$$

where v_c is the Earth's circular speed.

Evaluate $g_\Lambda / g_{\rm Newt}$ numerically, and comment on the size of the effect Λ has, and the use of solar system results to constrain Λ.

3. *Evidence for Acceleration: Luminosity Distances.*

   (a) **[1 point]** In class we saw that a standard candle, emitting at cosmic epoch (t_1, r_1) and detected here at $(t_0, 0)$ has a cosmic luminosity distance

   $$d_L = \frac{S_\kappa(r_1)}{a(t_1)} \tag{3}$$

   For the case of a flat universe ($\kappa = 0$), eliminate r_1 to show that d_L is purely a function of the emission epoch:

   $$d_L(a_1) = \frac{c}{H_0}\frac{1}{a_1}\int_{a_1}^{1} \frac{da}{\sqrt{\Omega_{\rm m} a + \Omega_\Lambda a^4}} \tag{4}$$

   $$d_L(z_1) = \frac{c}{H_0}(1+z_1)\int_0^{z_1} \frac{dz}{\sqrt{\Omega_{\rm m}(1+z)^3 + \Omega_\Lambda}} \tag{5}$$

   You may ignore the contribution from radiation.

   Finally, show that for small $z \ll 1$, $d_L(z)$ gives the usual Hubble's law expression for distance in a Newtonian cosmology.

   (b) **[2 points]** Find the exact expression for $d_L(z)$ for a flat universe filled with a substance having an equation of state with constant w.

   Use this general result to find the expression for $d_L(z)$ in an Einstein-de Sitter universe with $\Omega_{\rm m} = 1$. Then find $d_L(z)$ for a flat universe that has no acceleration ($w = -1/3$) and finally for a universe with $\Omega_\Lambda = 1$ and $\Omega_{\rm m} = 0$.

   Compare the results at $z = 0.1, 1, 10$? Comment on the implications for how these be distinguished observationally.

4. *Cosmic Acceleration.* Supernova luminosity distance data seem to require and accelerating universe. We now explore the consequences of this result.

   (a) **[1 point]** In class we considered a universe with a single component ρ_w with equation of state parameter $P = w\rho$, with w a constant. We saw that for such a universe to accelerate, it is a necessary (but not sufficient!) requirement that $w < -1/3$.

   Now consider a universe possibly like ours, with both this dark energy, but also matter, with density parameters $\Omega_w = 0.7$ and $\Omega_{\rm m} = 0.3$ (you may ignore curvature and radiation). Explain why the $w < -1/3$ condition is not sufficient for acceleration in such a universe. Then find the limit on w which will allow acceleration in such a universe; you should find that the revised limit makes w more negative. Does your result rule out or allow a cosmological constant?

   (b) **[1 bonus point]** In the case of a cosmological constant, an instructive diagram plots Ω_Λ versus $\Omega_{\rm m}$. In such a plot, there is region for $\Omega_\Lambda > 1$ which extends an increasing distance from the Ω_Λ axis. This region is often given the provocative label, "*no big bang.*" To make sense of this, consider a point in this region. Show that the Friedmann equation does indeed imply that for such a universe, the singularity $a = 0$ can never be achieved. *Hint:* pick a simple $(\Omega_{\rm m}, \Omega_\Lambda)$ pair from this region, and remember that in general $\Omega_0 \neq 1$ in this region.

5. *The Dark Energy Survey.* To get a better handle on Dark Energy, a large effort at Illinois (and elsewhere) is focused on the Dark Energy Survey (DES), and observational effort which uses different, complementary methods to get at the dark energy parameter w and to start to measure its possible evolution dw/dz.

   (a) **[1 point]** One aspect of the DES is to measure the luminosity distances to just under 2000 Type Ia supernovae in the redshift range $0.3 < z < 0.75$; this will become the recordholder for the largest such sample. Consider a set of typical DES Type Ia supernovae all at $z \approx 0.5$. Following the result in part (b) of Problem 3, find the ratio of matter- and Λ-dominated luminosity distances at this redshift.

   A typical distance measure ("distance modulus") $m = 5 \log_{10}(d_L/10 \text{ pc})$ can be determined to an accuracy of $\delta m \simeq 0.15$ magnitudes. Will one event be able to solidly distinguish the matter-only, no-acceleration and Λ-only cases (that is, how does $m_{\rm m} - m_\Lambda$ compare with δm)? Will it be possible to distinguish between the $w = -0.9$ and $w = -1$ cases?

   (b) **[1 point]** A major goal of the DES is to discover and measure the masses of all galaxy clusters in a $\Omega = 4000$ square degree region, out to a redshift $z_{\rm max} = 1.3$. The survey will this report the number $N(z)$ of clusters as a function of redshift out to this large distance. Note that this is somewhat like your exercise in Problem Set 1 of finding galaxies out to a limiting redshift.

   The number of observed clusters evolves as a function of redshift. Part of this is due to the fact that the cluster number density itself evolves as structures grow; we will return to this effect later. For now, we will focus on the fact that even if clusters have a constant comoving density, the discovery function $N(z)$ changes simply due to the changing cosmic volume per redshift bin.

   Compute the cosmic comoving volume change per unit redshift and solid angle, $d^2V_{\rm com}/dz\, d\Omega$, for a FLRW universe. In particular, show that the result can be written as

   $$\frac{d^2V_{\rm com}}{dz\, d\Omega} = \frac{c}{H(z)} \frac{d_L(z)^2}{(1+z)^2} \tag{6}$$

   where d_L is the luminosity distance.

   (c) **[1 point]** We immediately see that the comoving volume depends on cosmology manifestly through $H(z)$ as well as through d_L. For the three universes in Problem 3(b), compare the comoving volume in the survey solid angle $\Delta\Omega_{\rm DES} \simeq 1$ sr and a redshift bin $\Delta z = 0.1$ centered on $z = 1$. If the comoving cluster density at these redshifts is $n_{\rm cluster} \sim 3 \times 10^{-6}\ {\rm Mpc}^{-3}$, what should be the difference in the observed number $N_{\rm obs}$ of clusters in these two cosmologies? Given that real cluster samples will have statistical fluctuations ($\delta N_{\rm stat} \sim \sqrt{N_{\rm obs}}$), could DES distinguish these cases?