Astronomy 596/496 PC   Spring 2010
Problem Set #4

Due in class: Friday, March 19
Total points: 10+2

Throughout this problem set, whenever a cosmological parameters need to be chosen, we will adopt the concordance cosmology, i.e., a flat $\Omega_{\rm m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$ universe with $\eta = 6 \times 10^{-10}$.

ASTR496PC Students: You may drop choose to drop all of any numbered problem (i.e., all of its sub-parts if it has them).

1. *Redshift Evolution in Real Time as a Probe of Cosmic Expansion History.* In class we showed that redshifts are related to the cosmic scale factor at photon emission and observation via
$$z = \frac{a(t_{\rm obs})}{a(t_{\rm em})} - 1 \tag{1}$$
where $a_{\rm obs} = 1$ for present-epoch observations of interest to us. One usually thinks of the redshift of an object at fixed comoving distance r as a fixed measure equivalent to r, and/or a fixed measure of the emission epoch. While this is true for most practical purposes, it is not strictly correct. Since the 1960's work of Alan Sandage[1] and UIUC's own George McVittie[2] , it has been known that the time *change* of redshifts pose a potentially powerful test of cosmology generally and of cosmic acceleration (and hence dark energy) particularly.

   (a) **[1 point]** Starting with eq. (1), derive the McVittie equation for the observed evolution of redshift for an object
$$\frac{dz}{dt_{\rm obs}} = (1+z)\, H(t_{\rm obs}) - H(t_{\rm em}) = (1+z)\left[1 - \frac{1}{1+z}\frac{H(t_{\rm em})}{H_0}\right]\frac{1}{t_{H,0}} \tag{2}$$
   where $H(t)$ is the expansion rate evaluated at (and observed at) time t.

   (b) **[1 point]** Show that $dz/dt_{\rm obs} = 0$ for an "coasting" universe which has no acceleration, i.e., an expansion with $\ddot{a} = 0$. This implies that $dz/dt_{\rm obs}$ is a probe of cosmic acceleration/deceleration.

   For a matter-only universe show that $dz/dt_{\rm obs} < 0$, while for a Λ-dominated universe show that $dz/dt_{\rm obs} > 0$. Interpret these results physically.

   (c) **[1 point]** If we monitor the spectrum of an object at fixed comoving distance over a time δt, then the wavelength $\lambda_{\rm obs} = (1+z)\lambda_{\rm em}$ of spectral feature will drift by a fractional amount $\delta\lambda_{\rm obs}/\lambda_{\rm obs} = \lambda_{\rm em}/\lambda_{\rm obs}\; dz/dt_{\rm obs}\;\delta t = (1+z)^{-1} dz/dt_{\rm obs}\;\delta t$, equivalent to a Doppler velocity drift of $\delta v = c\delta z/(1+z)$.

[1]Sandage, A. 1962, ApJ, 136, 319. This paper also calculates the time evolution of the *luminosity* of a source at fixed comoving distance. I leave it to the reader to see why this would be an even more difficult thing to measure than the time-change of redshift.

[2]McVittie, G. C. 1962 ApJ, 136, 334. Somewhat oddly, this is a separately-authored appendix to the Sandage (1962) paper, in which McVittie extends Sandage's analysis for general combinations of $\Omega_{\rm m}$ and Ω_Λ.

Sensitive spectroscopic techniques have been developed to find planets via the small change in the Dopper shift of the parent star due to the gravitational influence of the planet. Current methods can detect velocity changes down to about $\delta v_{\rm obs} \sim 1$ m/s over timescales as long as $\delta t \sim 10$ yr.

Using the above results, find the Doppler velocity drift of a $z = 3$ object over a timescale of $\delta t = 10$ yr, in a $\Omega_{\rm m} = 0.3$ and $\Omega_\Lambda = 0.7$ cosmology. Can this be observed with current techniques? What complications might make this measurement and its interpretation difficult? (*Hint*: real objects at $z = 3$ are not point sources, and do have internal motions.) How might some of these difficulties be overcome?

Such an observational campaign is sometimes known as the Sandage-Loeb[3] test, which has been known of for decades (thanks to Sandage and McVittie) but has received a revival of accelerated interest recently[4].

(d) **[1 bonus point]** Imagine it is (or becomes) possible to make reliable measurements of redshift drifts over a substantial redshift range, say $z = 0.5 - 3$. Explain how such measurements could be used to test cosmology in general, and dark energy models in particular.

2. *Cosmic Thermal Photodissociation.*

(a) **[1 bonus point]** In several cosmic situations we will want to know the number density of thermal photons (or other relativistic particles) with energies exceeding some scale ϵ which lies above the particles' temperature T. That is, we are interested in the number density of photons in the "high-energy tail." Show that, for $\epsilon \gg T$, the number density of particles with energies above ϵ is

$$n_{\rm rel}(> \epsilon) \approx \frac{g}{2\pi^2} \epsilon^2 T e^{-\epsilon/T} \tag{3}$$

Hint: in the integral over phase space, it may be useful to change variables from p to $q = p - \epsilon$.

Finally, eq. (3) is written in units where $\hbar = c = k_{\rm Boltz} = 1$. Revise eq. (3), replacing these factors as needed to restore the correct observable units for n.

(b) **[1 point]** We typically will want to use eq. (3) to compute the number of high-energy, dissociating photons per baryon. In particular, it is of interest to find when $n_{\rm rel}(> \epsilon)$ drops below the baryon number density $n_{\rm b}$. Find an expression which shows how the temperature at which this occurs, $T_{\rm dis}$, is related to the energy scale ϵ and the baryon-to-photon ratio η.

Give an approximate expression for $T_{\rm dis}$, ignoring logarithmic corrections (i.e., ignore terms like $\ln T$).

Finally, consider the case in which ϵ is the energy needed for photons to break up, or "dissociate," a bound state of baryons. Interpret the physical significance $T_{\rm dis}$ and explain why it is not just ϵ.

[3] Loeb, A. 1998, ApJL, 499, L111

[4] See, e.g., Lake, K. 1981, ApJ, 247, 17; Lake, K. 2007, PRD, 76, 063508; Balbi, A., & Quercellini, C. 2007, MNRAS, 382, 1623; Uzan, J.-P., Bernardeau, F., & Mellier, Y. 2008, PRD, 77, 021301;

3. *The Epoch of Recombination.*

   (a) **[1 point]** Estimate the temperature of recombination, using your result from question 2b above. Be sure to explain your choice of energy scale ϵ. Go on to estimate the redshift $z_{\rm rec}$ of recombination.

   (b) **[1 point]** To make a more refined estimate, calculate the redshift $z_{\rm rec}$ of recombination, using the Saha equation. Take as the condition for recombination that only a fraction $X_{e,{\rm rec}} = 10\%$ of the electrons remain free.

   Compare your result to the estimate from part 3(a), and comment.

   (c) **[1 point]** Show how your result from part (b) would change if you had defined recombination by free electron fractions of $X_{e,{\rm rec}} = 50\%$, or $X_{e,{\rm rec}} = 1\%$. Comment on the result and give your estimate of the quantitative uncertainty or "fuzziness" $\Delta z_{\rm rec}$ in the recombination redshift due to this arbitrariness in assigning a unique instant to this continuous (but rapid) event.

4. *Angular Diameter of the Recombination Horizon.*

   (a) **[1 point]** Find an expression for the angular diameter distance to an object whose redshift z falls within the matter-dominated epoch of a concordance cosmology. Evaluate the result numerically for $z = z_{\rm rec}$.

   Then find an expression for the physical (i.e., not comoving) size of the particle horizon for an instant whose redshift z falls within the matter-dominated epoch of a concordance cosmology. Evaluate the result numerically for $z = z_{\rm rec}$.

   (b) **[1 point]** Given the results for parts (a) and (b), calculate an expression for the angular diameter $\theta_{\rm hor,rec}$ of the horizon at recombination in terms of $z_{\rm rec}$. Evaluate your result numerically and express it in degrees.

   Finally, interpret your result physically: on the basis of your calculations (i.e., don't worry yet about inflation) how would you understand CMB temperature differences (or sameness) observed on angular scales smaller than $\theta_{\rm hor,rec}$? larger than $\theta_{\rm hor,rec}$?

5. **[1 point]** *Olber's Paradox Revisited.* In PS2 you found the resolution to Olber's paradox as it concerns starlight. But unbeknownst to Olber, there is an entirely different population of cosmic photons, namely the thermal CMB.

   Find the temperature $T_{\rm ls}$ of the CMB at last scattering (which you can take to be the same as recombination), and from this the expression for *surface brightness* $I_{\rm cmb}$ as seen by an observer then. The CMB encounters essentially no absorption after last scattering, and so in a static universe the surface brightness would be conserved. However, in an expanding universe the surface brightness of an object at redshift z *decreases* as $I_{\rm obs}(z) = I_{\rm em}/(1+z)^4$. Explain this $(1+z)^4$ factor either by considering the evolution of the CMB temperature, or by generally considering the evolution of the ingredients of surface brightness (angular area and flux), or both.

   Finally, find the surface brightness of the CMB today, compare it to the surface brightness of the Sun (which you found in PS2). Comment on how your results modify our understanding of Olber's paradox and its resolution. Also comment on the future CMB appearance in an increasingly dark-energy-dominated universe.