Astronomy 596/496 PC Spring 2010
Problem Set #6
Due in class: **Monday**, April 12
Total points: 10+2

ASTR496PC Students: You may drop choose to drop all of any numbered problem (i.e., all of its sub-parts if it has them). You may *also* choose to substitute Problem 6 for one of the other numbered problems. That is, you are responsible for 5 numbered problems, of which one may be Problem 6.

1. *Observational Requirements for Inflation: Cosmic e-foldings.*

   (a) **[1 point]** Imagine that we have information which tells us that the universe pass through an cosmic epoch in either the radiation or matter eras. It follows that the universe didn't recollapse or go to zero density soon thereafter, and that we have nearly flat universe today, so that the curvature then must have been small. Given some epoch z, and the current limits $\|\Omega_{\kappa,0}\| \equiv \|\Omega_0 - 1\| \leq 0.01$, find and expression for the limits on the curvature parameter $\|\Omega_\kappa\| \equiv \|\Omega(z) - 1\|$. Note that the results are different depending on whether the epoch is matter- or radiation-dominated.

   One way to state the flatness problem is that "generically" one expects the curvature term comparable to the others: $\|\Omega - 1\| \sim 1$, while you have found $\|\Omega - 1\| \ll 1$. Use your result to deduce the required number $N_{\rm min}$ of inflationary e-foldings prior to the epoch z in order to leave it as flat as you have required. To do this, assume that prior to inflation, the generic condition $\|\Omega - 1\| \sim 1$ held. Then you can calculate how much the curvature would need to be inflated to meet some observed bound on $\|\Omega - 1\|$.

   (b) **[1 point]** Apply your result from (a) to find $N_{\rm min}$ as implied by these cosmic epochs: recombination, BBN, the "Fermilab era" when $T \sim E_{\rm Tevatron} = 1$ TeV, the GUT era $\sim 10^{15}$ GeV, and the Planck epoch.

   (c) **[1 bonus point]** (Following Liddle & Lyth 3.5) If the universe underwent a GUT transition $T \sim 10^{15}$ GeV), it is expected that one magnetic monopole ($m \sim 10^{15}$ GeV) was created per Hubble volume. In the absence of inflation, compute the relic mass density of monopoles today; you should get an uncomfortably large number. Using the limit $\Omega_{\rm Monopole,0} \lesssim 10^{-6}$ (Parker bound), compute require the number of e-foldings of inflation needed to respect this bound. Compare your result to those above, and comment.

2. *Scalar Field Dynamics.* A classical and spatially homogeneous scalar field ϕ which only interacts with itself (via a potential V) and with gravity has an equation of motion in a FRW universe given by

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0 \qquad (1)$$

   (a) **[1 point]** In a non-expanding universe, show that the equation of motion implies that $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ is a constant.

   Find an expression for $\dot{\rho}_\phi$ in an expanding universe, and interpret it physically.

(b) **[1 point]** Show that if the kinetic term dominates the ϕ energy density (i.e., if V is negligible), $\rho_\phi \propto a^{-6}$. Also find the value of w_ϕ in this case.

If the kinetic term in ρ_ϕ not only dominates V but also the rest of the energy density in the universe ("kination"), go on to find the time evolution $\phi(t)$.

It is not known if the universe ever underwent such a phase. Comment on why such a phase is unsuitable for inflation.

3. *Slow-Roll Conditions.*

(a) **[1 point]** Show that the slow-roll requirements that $\dot{\phi}^2/2 \ll V(\phi)$ and $\ddot{\phi} \ll 3H\dot{\phi}$ are equivalent to the statements that

$$\epsilon(\phi) \equiv \frac{m_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \quad (2)$$

$$\|\,\eta(\phi)\,\| \equiv m_{\rm Pl}^2 \left\|\frac{V''}{V}\right\| \ll 1 \quad (3)$$

where η here is neither the baryon-to-photon ratio nor conformal time! Also, here and throughout, we follow Liddle & Lyth in using the "reduced Planck mass" $m_{\rm Pl} = M_{\rm Pl}/\sqrt{8\pi} = \sqrt{\hbar c/8\pi G}$, so that, e.g., Friedmann reads $H^2 = \rho/3m_{\rm Pl}^2$.

(b) **[1 point]** Show that if both ϵ and η are strict constants, independent of ϕ, this uniquely specifies the inflation potential to be of the form

$$V(\phi) = V_0 e^{\phi/\mu} \quad (4)$$

Find the value of the energy scale μ in terms of ϵ, η, and physical constants.

4. *A Quartic Potential.* (Following Liddle & Lyth 3.7). Consider the case where the scalar field has a quartic potential: $V(\phi) = \lambda\phi^4$, with λ a dimensionless constant.

(a) **[1 point]** In the slow-roll approximation, with initial value $\phi = \phi_i$ and $a = a_i$ at $t = t_i$ show that the field decays exponentially for $t > t_i$:

$$\phi(t) = \phi_i \, \exp\left[-\sqrt{\frac{32\lambda m_{\rm Pl}^2}{6}}(t - t_i)\right] \quad (5)$$

Use this solution to find $\rho_\phi(t)$, and comment on the implications of your result.

(b) **[1 point]** Go on to show that, still within the slow-roll approximation, that the scale factor follows a double exponential form

$$a(t) = a_i \exp\left(\frac{\phi_i^2}{8m_{\rm Pl}^2}\left\{1 - \exp\left[-\sqrt{\frac{64\lambda m_{\rm Pl}^2}{3}}(t - t_i)\right]\right\}\right) \quad (6)$$

(c) **[1 point]** Show that initially, i.e., for small $t - t_i$, the expansion is exponential. Calculate the time constant ξ (from $a \sim e^{\xi t}$) and demonstrate that it equals the (slow-roll) Hubble parameter during inflation.

(d) **[1 bonus point]** Find the value(s) of ϕ when the slow-roll conditions first break down. Do they break down at the same place?

5. *Inflation and Dark Energy.* Inflation and the quintessence model for dark energy share many similarities, but often the theories are discussed using different language: inflation focusses on the slow-roll condition and parameters ϵ and η, while dark energy focusses on the equation-of-state parameter w. A few weeks ago, an elegant paper (Ilic, Kunz, Liddle, & Frieman 2010, `arXiv:1002.4196`) spelled out the connection between these two parameterizations.

   (a) **[1 bonus point]** Show that, when the slow-roll conditions are satisfied, then

   $$1 + w = \frac{2}{3}\epsilon \tag{7}$$

   valid to first order in the small slow-roll parameters.

   (b) **[1 point]** Use eq. (7) to infer that the equation of state of a slowly rolling inflationary period is similar to that of a universe dominated by dark energy.

6. *The Planck Mass.* Note: *only* ASTR496PC can receive credit for this problem by swapping it for one of the other assigned problems. ASTR 596PC students *will not* receive credit for this problem.

   (a) **[1 point]** In (special) relativity, a particle of mass m has a characteristic energy scale, mc^2, associated with it, and an characteristic momentum scale mc. In quantum mechanics there is a natural length scale associated with a particle of momentum p, namely the de Broglie wavelength $\lambda \sim \hbar/p$. One thus arrives at a lengthscale associated with relativistic quantum effects, namely the Compton wavelength $\lambda_c = \hbar/mc$. At length scales at or smaller than this, one expects relativistic quantum effects to be important, and in fact this is how we calculated the range of forces mediated by massive bosons (e.g., the weak force).

   In (classical) General Relativity, there is a natural length scale associated with a body of mass m, namely $r_{\rm gr} \sim Gm/c^2$ (this is half of the so-called Schwarzschild radius) . For the known fundamental particles, it turns out that $r_{\rm Sch} \ll \lambda_c$, which means that one may ignore General Relativity when describing them. However, if a particle had a mass m such that $r_{\rm Sch} = \lambda_c$, one would need a full General Relativistic quantum theory to describe the particle–i.e., *quantum gravity*. This mass scale is known as the Planck mass $M_{\rm Pl}$. Calculate $M_{\rm Pl}$ and give its value in energy units of GeV. Also calculate the associated Planck time and Planck length. These correspond to the temperature, age, and size of the Universe before which quantum gravity effects must be included, and thus mark the extreme edge of applicability of current (non-quantum-gravity) theories.

   (b) **[1 point]** Verify that in natural units ($\hbar = c = k_{\rm Boltz} = 1$), a radiation-dominated universe has an expansion rate $H \sim T^2/M_{\rm Pl}$.