Astronomy 596 PC   Spring 2006
Problem Set #7: The Final Frontier

Due in the instructor's mailbox (or hands!) on or before: Thursday, May 13, noon
Total points: 10+2

1. *Peculiar Velocities: Linear Analysis.*

   (a) **[1 point]** Consider the linearized analysis of perturbations in a non-expanding homogeneous Newtonian fluid at rest. Verify that the linearized equations can be combined to give the wave equation

   $$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta \tag{1}$$

   You do should only use the linearized fluid equations and not already assume plane wave solutions.

   (b) **[1 bonus point]** Now consider the velocity perturbation $\vec{u}$, still in a non-expanding fluid. We may decompose this vector into two part $\vec{u} = \vec{u}_{\|} + \vec{u}_{\perp}$, where $\vec{u}_{\|}$ has nonzero divergence and zero curl, and $\vec{u}_{\perp}$ has zero divergence and nonzero curl. Show that the curl of the velocity perturbation (sometimes called the vorticity $\vec{\omega} = \nabla \times \vec{u} = \nabla \times \vec{u}_{\perp}$) is constant in time, and use this to argue that the "rotational" velocity components with $\vec{k} \cdot \vec{u} = 0$ are constant as well. Comment on the physical reason for this (hint: what is the nature of the forces involved?).

   (c) **[1 point]** Still in a non-expanding fluid, focus on the curl-free velocity component; call this $\vec{u}_{\|}$. Since this field has no curl, it can be written as a gradient $\vec{u}_{\|} = \nabla \psi$, where ψ is known as the "velocity potential." Use the linearized fluid equations to show that ψ also obeys ths same wave equation as that governing δ. What do you infer about the behavior of $\vec{u}_{\|}$? Interpret your result physically. What is the behavior of $u_{\|}$ in the unstable regime?

   (d) **[1 bonus point]** Now consider cosmological perturbations in an expanding universe, and focus on the velocity. Making the same velocity decomposition, show that the curl of the velocity perturbation (sometimes called the vorticity $\vec{\omega} = \nabla \times \vec{u} = \nabla \times \vec{u}_{\perp}$) evolves with time as

   $$\omega \propto a^{-2} \tag{2}$$

   and thus the vorticity (and $\vec{u}_{\perp}$) redshifts away.

   (e) **[1 point]** Now consider only the curl-free peculiar velocity component; for brevity just call it $\vec{v}$. Show that we can combine the linearized continuity equation, and the fact that (for unstable modes) δ evolves with the linear growth factor $\delta(t) = D(t)\delta_0$, to infer

   $$\frac{1}{a} \nabla \cdot \vec{v} = -\frac{\dot{D}}{D} \delta \equiv -f(t)\, H(t)\, \delta \tag{3}$$

where the dimensionless factor

$$f(t) = \frac{\dot{D}/D}{\dot{a}/a} \quad (4)$$

encodes the relative rates of perturbation growth versus the growth of the scale factor. What is $f(t)$ for a matter-dominated universe?

Go on to use the linearized Gauss' theorem for the peculiar gravitational acceleration

$$\frac{1}{a}\nabla \cdot \vec{g} = -4\pi G\rho\delta \quad (5)$$

to infer that

$$\vec{v} = \frac{2f}{3H\Omega_m}\vec{g} \quad (6)$$

Interpret this result physically.

(f) **[1 point]** In light of eq. (6), consider the observed CMB dipole. What is the significance of its magnitude and direction? Given the magnitude $v_{\rm pec} \sim 600$ km/s, what is our peculiar acceleration with respect to the CMB frame, both in cm/s^2 and Mpc/Gyr^2? Comment on the result.

Describe an observational strategy to relate our peculiar velocity to the observed local matter distribution (i.e., nearby clusters and superclusters).

2. *Lyman-α Forest and Reionization of the Intergalactic Medium.*

(a) **[1 point]** Show that, in a matter-dominated universe, the optical depth for Lyα absorption (with cross section $\sigma \sim 8 \times 10^{-18}$ cm^2) out to some redshift z is

$$\tau_{\rm HI}(z) = \frac{2}{3\Omega_{\rm m}^{1/2}}\sigma n_{\rm HI}\frac{c}{H_0}(1+z)^{-3/2} \quad (7)$$

where $n_{\rm HI}$ is the (assumed constant) comoving density of neutral hydrogen.

(b) **[1 point]** In reality, $\tau_{\rm HI}$ varies wildly over every line of sight towards a quasar, due to the Lyα forest. This of course indicates that the real universe has a lumpy distribution of neutral hydrogen. Nevertheless, if we note that about half (or, say, $1/e$) of a typical high-redshift quasar's flux is carved out by the forest, we conclude that a sort of "ensemble average" optical depth is something like $\langle\tau_{\rm obs}\rangle \sim 1$ at redshift $z \sim 3$. Use this value, and your result from part (a) to find $\Omega_{\rm HI}(z)$.

Compare this with the cosmic baryon density to find the mean IGM neutral hydrogen fraction $1 - X_e = n_{\rm H}/n_{\rm baryon}$ as a function of z.

(c) **[1 point]** At $z = 3$ you should find $1 - X_e \sim 10^{-6}$ or so. Use your result and the Saha equation to estimate the temperature $T_{\rm IGM}$ needed to keep the IGM in equilibrium at this ionization. Express your answer in eV.

Go on to calculate the average thermal energy per baryon, and the comoving cosmic ionizing energy density in eV/cm^{-3}. This gives a feel for the required ionizing energy injection to reionize the universe. Comment on the result.

(d) **[1 point]** A massive star destined to become a supernova emits copious amounts of ionizing radiation. Take a typical massive star to have $M = 20M_\odot$, emits ionizing photons at a rate $Q_{\rm ioniz} \simeq 10^{49}\ {\rm s}^{-1}$ over a $\tau \sim 10^6$ yr lifetime. For simplicity, take these all to be at 13.6 eV.

Given this, and your results from the previous question, what is the fraction of baryons needed to go into $20M_\odot$ stars in order to reionize the universe. Comment on whether this seems plausible.

3. *The Cosmic Star-Formation Rate.* Measuring and understanding the history of cosmic star formation is a major topic in cosmology today. As discussed in class, the cosmic star formation rate $\dot{\rho}_\star(z)$ is now fairly well-determined out to redshifts $z \sim 2$.

(a) **[1 point]** A famous recent evaluation of the cosmic star formation rate appears in Hopkins & Beacom (2006, ApJ 451, 142), linked in the final lecture webpage. Consult Hopkins & Beacom's Figure 1 and the surrounding discussion, and find the value $\dot{\rho}_\star(0)$ of the cosmic star-formation rate today, i.e., at $z = 0$. To see if this makes sense, consider the quantity $\bar{\psi} = \dot{\rho}_\star(0)/n_{\rm gal}$, where $n_{\rm gal}$ is a measure of the number density of galaxies at $z = 0$, for example as you found in Problem Set 1.

Explain why $\bar{\psi}$ should be a measure of the average star-formation rate of a typical galaxy today.

Then evaluate $\bar{\psi}$ using the Hopkins & Beacom value for $\dot{\rho}_\star(0)$, and $n_{\rm gal}$ you found in Problem Set 1 (or from class notes). Compare your result to the Milky Way star-formation rate $\psi_{\rm MW} \simeq 1\ M_\odot/{\rm yr}$ can comment.

(b) **[1 point]** Out to redshift $z \sim 1$, the cosmic star-formation rate grows roughly as $\dot{\rho}_\star \propto (1+z)^3$, so that $\dot{\rho}_\star(z) = (1+z)^3\ \dot{\rho}_\star(0)$. Assuming this dependence, integrate this rate over cosmic time, i.e., find $\rho_{\rm sf} = \int_{t(z=1)}^{t_0} \dot{\rho}_\star\ dt = \int_{a(z=1)}^{1} \dot{\rho}_\star\ da/(aH)$, using the expansion rate for a matter-dominated universe with $\Omega_{\rm m} = 0.3$. Also find $\Omega_{\rm sf}$. What physically should $\rho_{\rm sf}$ and $\Omega_{\rm sf}$ measure?

Compare your results with the baryon density parameter $\Omega_{\rm baryon}$, and the density parameter for stellar luminous matter $\Omega_{\rm lum}$ found in Problem Set 1 and in class notes. Comment on the results.