Today's ASTR 596/496 Cosmo Café Special:
Relativisitic Gastrophysics!

- Get a gut feeling for cosmic geometry!

- All three tasty possibilities available:
  - ▷ **flat**
  - ▷ **positively** curved
  - ▷ **negatively** curved

- Try 'em all!

Bon appetit!

Astro 596/496 PC
Lecture 10
Feb. 10, 2010

Announcements:
- PF2 due Friday noon

Last time: Relativistic Cosmology
cosmic spacetimes: maximally symmetric
Q: fundamental observers?
Q: comoving coordinates?
Q: cosmic time?

Today:
- cosmic geometry
- Friedmann-Lemaître-Robertson-Walker (FLRW) metric
- physics in a FLRW universe

Cosmological Principle and Cosmic Spacetime

Executive Summary

Cosmo Principle $\rightarrow$ at any time, space is **maximally symmetric**

- strongly restricts allowed spacetime structure
- there exist a set of **fundamental observers** (FOs)
  (or "frames" or "coordinate systems")
  who see U as homogenous and isotropic
- FOs "ride on" or are at rest w.r.t. **comoving coordinates**
  which don't change with expansion
  but do of course physically move apart
- FO clocks all tick at same rate, measure **cosmic time** t

Note: in a generic spacetime, not possible to "synchronize clocks"
in this way

Cosmological spacetime encoded via cosmic **metric**
which determines how the interval depends on coordinates
any observer computes interval between events as
$ds^2 =$ (elapsed time)^2 – (spatial displacement)^2

Cosmic metric so far:

$$ds^2 = dt^2 - g_{ii}(dx^i)^2 \tag{1}$$

where: t is cosmic time

now impose *isotropy*

- at any cosmic t, interval invariant under rotations
- pick arbitrary origin, then (comoving) spherical coords
  the usual r, θ, ϕ, with $r^2 = x^2 + y^2 + z^2$
  and arbitrary origin (usually, but not always, here!)

Q: now that does metric look like?

For *fundamental* observers, maximal symmetry demands metric which can* be written as:

$$ds^2 = dt^2 - a(t)^2 d\ell^2_{\rm com} \quad (2)$$

$$= dt^2 - a(t)^2 \left[f(r) dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (3)$$

$a(t)$ is the cosmic scale factor

$f(r)$ is as yet undetermined

- for flat (Euclidean) space, $f(r) = 1$
- so $f \neq 1 \rightarrow$ non-Euclidean spatial geometry = curved space!

Q: why same time dep for radial and angular displacements?

Note power of cosmo principle

$\rightarrow$ only allowed dynamics is uniform expansion $a(t)$!

*other space & time coordinates possible and sometimes useful

but in all cases space and time must *factor* in this way

Curvature

maximal symmetry requires that Universe spatial "3-volume"
is a "**space of constant curvature**"

at any time t: cosmic curvature is a length $\mathcal{R}(t)$

- today: $\mathcal{R}(t_0) \equiv R$
- *Q: dependence on scale factor?*

For the relatvists: max symmetry means *spatial* curvature tensor must take the form

$$R^{(3)}_{ijk\ell} = \frac{\kappa}{\mathcal{R}(t)^2} \left(h_{ik} h_{jl} - h_{jk} h_{il} \right) \tag{4}$$

where $\kappa = -1$, 0, or $+1$
and h is the spatial part of metric g

Note: the curvature scalar is really one single number K

but for $K \neq 0$ one can identify a sign $\kappa \equiv K/\|K\|$ and lengthscale $\mathcal{R}^2 \equiv 1/\|K\|$

Spaces of Constant Curvature

Amazing mathematical result:
despite enormous constraints of maximal symmetry
GR does *not* demand cosmic space to be flat (Euclidean)
as assumed in pre-relativity and special relativity

GR allows *three* classes of cosmic spatial geometry
each of which is a space of constant (or zero) curvature
- positive curvature → hyper-spherical
- negative curvature → hyperbolic
- zero curvature → flat (Euclidean)

`www: cartoons`

All of these are *allowed* by GR and maximal symmetry
but *our* universe can have only *one* of them
Q: how do we know which of these our U has "chosen"?

Positive Curvature: A (Hyper-)Spherical Universe

to get an intuition: consider ordinary sphere ("2-sphere")
using coordinates in Euclidean space ("embedding")
sphere defined by $(x, y, z) \in x^2 + y^2 + z^2 = R^2 = const$

Coordinates on the sphere:

- usual spherical coords: center, origin outside of the space
- alternative coordinates: origin in the space

  `www: artist's sketch`

[origin]: at north pole $z = +R$, $x = y = 0$

[r] distance from z axis $\Leftrightarrow$ latitudes

$r^2 = x^2 + y^2 = R^2 - z^2$

[θ] angle from x axis $\rightarrow$ longitudes

∞ [$R\chi$] arclength on sphere from pole

$\rightarrow$ χ is usual spherical angle from pole

2-sphere metric:

in 3-D embedding space: $d\ell^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2$

but points, intervals constrained to lie on sphere:

$R^2 = r^2 + z^2 = const$

$d(R^2) = 0 = xdx + ydy + zdz = rdr + zdz$

so $dz = -rdr/z \rightarrow$ can eliminate z

thus in polar coords with origin at N Pole

$$d\ell^2 = dr^2 + r^2 d\theta^2 + dz^2 = \left(1 + \frac{r^2}{R^2 - r^2}\right) dr^2 + r^2 d\theta^2 \quad (5)$$

$$= \left(\frac{R^2}{R^2 - r^2}\right) dr^2 + r^2 d\theta^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 \quad (6)$$

not Euclidean expression!

curved space: curvature $R^2 = const$!

Exploring Sphereland

coordinates for (2-D) observers on sphere, centered at N Pole:

$$d\ell^2 = d\ell_r^2 + d\ell_\theta^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 = R^2 d\chi^2 + R^2 \sin^2\chi \, d\theta^2$$

N Pole inhabitant (2-Santa) measures radial distance from home:
$d\ell_r = dr/\sqrt{1 - r^2/R^2} \equiv R d\chi$
$\rightarrow$ radius is $\ell_r = R \sin^{-1}(r/R) \equiv R\chi$

Example: construct a *circle*
locus of points at same radius ℓ_r

- circumference $dC = d\ell_\theta = r d\theta = R \sin\chi d\theta$
  $\rightarrow C = 2\pi R \sin\chi < 2\pi\ell_r$
- area $dA = d\ell_r d\ell_\theta = R^2 \sin\chi \, d\chi d\theta$
  $\rightarrow A = 2\pi R^2 (1 - \cos\chi) < \pi\ell_r^2$

Q: why are these right?

3-D Life in a 4-D Sphere

generalize to 3-D "surface" of sphere in 4-D space ("3-sphere"), constant positive curvature R:
3-D spherical coordinates centered on "N pole"

spatial line element

$$d\ell^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{7}$$

- sky still has solid angle $d\Omega = \sin\theta d\theta d\pi$, $\int d\Omega = 4\pi$
- radial (proper) distance $\Delta\ell_r = R \sin^{-1}(r/R) \equiv R\chi$
- so we have found, for $\kappa = +1$,
  RW metric has $f(r) = 1/(1 - r^2/R^2)$

Q: guesses for zero, negative curvature metrics?

Friedmann-Lemaître-Robertson-Walker Metric

Robertson & Walker:
maximal symmetry imposes metric form

Robertson-Walker line element (in my favorite units, coords):

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2/R^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right)$$

where cosmic geometry encoded via κ:

$$\kappa = \begin{cases} +1 & \text{pos curv: ``spherical''} \\ 0 & \text{flat: ``Euclidean''} \\ -1 & \text{neg curv: ``hyperbolic''} \end{cases} \tag{8}$$

Friedmann-Lemaître-Robertson-Walker Cosmology

Friedmann & Lemaître:
solve GR dynamics (Einstein equation)
for stress-energy of "perfect fluid" (no dissipation)

The Einstein Equation and Robertson-Walker

Einstein eq: $R_{\mu\nu} - 1/2\, R g_{\mu\nu} = 8\pi G T_{\mu\nu}$
derivatives in Einstein eq come from curvature tensor $R_{\mu\nu}$
$\rightarrow$ schematically: "$R \sim \partial^2 g \sim G\rho$" – like Newtonian Poisson eq
but the only undetermined function in the metric
is the scale factor a, which only depends on t:
so: Einstein eqs $\rightarrow$ ODEs which set evolution of $a(t)$
$\Rightarrow$ these are the Friedmann equations!
and: in RW metric, local energy conservation $\nabla_\nu T^{\mu\nu} = 0$
$\Rightarrow$ gives 1st Law: $d(\rho a^3) = -p d(a)^3$

More detail in today's Director's Cut Extras

Life in a FRLW Universe

FLRW metric + Friedmann eqs for $a(t)$
$\rightarrow$ all you need to calculate anything
particle motions, fluid evolution, observables...

Excellent first example: Propagation of light

We want to know
- photon path through spacetime
- evolution of photon λ, E during propagation
- detected redshift

Q: how to calculate these?
Q: relevant equations?
Q: coordinate choices?

Worked Example: Photon Propagation

photon path:
null trajectory $ds = 0$ (Fermat)

label events:
⋆ emitted at $r_{\rm em}$, $t_{\rm em}$
⋆ observed at $r_{\rm obs} = 0$, $t_{\rm obs}$

trajectory coordinates:
free to choose *purely radial* motion: $d\theta = d\phi = 0$
Q: why?

for radial photons, $ds = d\theta = d\phi = 0$

then FLRW metric gives (*Q: why $-$ sign?*):

$$dt = -a(t)\frac{dr}{\sqrt{1-\kappa r^2/R^2}} = -a(t)\ d\ell_{\rm com}$$

$$\frac{dt}{a(t)} = -\frac{dr}{\sqrt{1-\kappa r^2/R^2}} = -d\ell_{\rm com}$$

$$\int_{t_{\rm em}}^{t_{\rm obs}} \frac{dt}{a(t)} = \int_0^{r_{\rm em}} \frac{dr}{\sqrt{1-\kappa r^2/R^2}} = \Delta\ell_{\rm com}$$

relates $(t_{\rm em}, r_{\rm em})$ to $(t_{\rm obs}, r_{\rm obs})$

$\rightarrow$ this is photon trajectory!

Trajectories of Cosmic Photons

for FOs at $r_{\rm em}$ and $r_{\rm obs} = 0$,
any $t_{\rm em}$ and $t_{\rm obs}$ pairs have

$$\int_{t_{\rm em}}^{t_{\rm obs}} \frac{dt}{a(t)} \;=\; \int_0^{r_{\rm em}} \frac{dr}{\sqrt{1-\kappa r^2/R^2}}$$

time-dep time-indep

Since RHS is time-independent *Q: why?*
then *any* two pairs of emission/observation events
between comoving points $r \rightarrow 0$ must have

$$\int_{t_{\rm em,1}}^{t_{\rm obs,1}} \frac{dt}{a(t)} = \int_{t_{\rm em,2}}^{t_{\rm obs,2}} \frac{dt}{a(t)} \tag{9}$$

Director's Cut Extras For Relativists

Perfect fluid:

- "perfect" $\rightarrow$ no dissipation (i.e., viscosity)
- stress-energy: given density, pressure fields ρ, p and 4-velocity field $u_\mu \rightarrow (1,0,0,0)$ for FO

$$\begin{aligned} T_{\mu\nu} &= \rho u_\mu u_\nu + p(g_{\mu\nu} - u_\mu u_\nu) && (10) \\ &= \text{diag}(\rho, p, p, p)_{\text{FO}} && (11) \end{aligned}$$

Recall: stress-energy conservation is

$$\nabla_\nu T^{\mu\nu} = 0 \quad (12)$$

where ∇_μ is covariant derivative
For RW metric, this becomes:

$$d(a^3 \rho) = p d(a^3) \quad (13)$$

1st Law of Thermodynamics!

Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{14}$$

Given RW metric (orthogonal, max symmetric):

- *Q: how many nonzero Einstein eqs generally? here?*
- *Q: what goes into $G_{\mu\nu}$? what will this be for RW metric?*

Einstein eq:

$G_{\mu\nu}, T_{\mu\nu}$ symmetric 4×4 matrices $\rightarrow$ 10 independent components

in general, Einstein $\rightarrow$ 10 equations

but cosmo principle demands: space-time terms $G_{0i} = 0$

and off-diagonal space-space $G_{ij} = 0$

else pick out special direction $\Rightarrow$ only diagonal terms nonzero

and all 3 "p" equations same

Einstein → two independent equations

$$G_{00} = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\kappa}{R^2a^2} \quad (15)$$
$$= 8\pi G T_{00} = 8\pi G\rho \quad (16)$$
$$G_{ii} = 6\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\kappa}{R^2a^2} \quad (17)$$
$$= 8\pi G T_{ii} = 8\pi G p \quad (18)$$

After rearrangement, these become
the Friedmann "energy" and acceleration equations!