Astro 596/496 PC
Lecture 20
March 5, 2010

Announcements:

- PS3 due
- PF 4 out, due next Friday noon

Last time: theory of isotropic CMB spectrum
key aspect: Thompson scattering is *only* process acting
for most photons (i.e., for all photons with $h\nu \lesssim 40kT$)
Given a photon spectrum $I_\nu$ *prior* to decoupling
*Q: what is spectrum after Thompson freezeout?*
Observed (post-decoupling) CMB spectrum: *thermal*
*Q: implications?*

*Q: what physically controls onset of decoupling?*
*Q: naïve estimate of recombination* $T_{\rm rec}, z_{\rm rec}$*?*
*Q: zeroth-order treatment of free electron fraction* $X_e(T)$*?*

# Recombination: Improved Naïve View

Given H binding energy

$$B_{\rm H} = E(p) + E(e) - E({\rm H}) = 13.6 \text{ eV}$$

simple estimate of recomb epoch goes like this:

Binding sets energy scale, so
⋆ when particle energies above $B_{\rm H}$: U ionized,
⋆ otherwise: U neutral
→ naïvely expect transition at $T_{\rm rec,naive} = B_h \sim 150,000$ K

But we know $T = T_0/a$, so estimate recomb at

$$\left.\begin{aligned} a_{\rm rec,naive} &= \frac{T_0}{T_{\rm rec,naive}} &\sim 2 \times 10^{-5} \\ z_{\rm rec,naive} &= \frac{T_{\rm rec,naive}}{T_0} - 1 &\sim 50,000 \end{aligned}\right\} \text{wrong!}$$

*Q: guesses as to what's wrong?*
*Q: how to do this right?*

# Statistical Mechanics and Cosmology

For much of cosmic time contents of U. in *thermal equilibrium*

*statistical mechanics*: at fixed$T$ $\rightarrow$ matter & radiation $n, \rho, P$
then cosmic $T(a)$ evolution $\rightarrow$ $n, \rho, P$ at any epcoh

Boltzmann: consider a particle (elementary or composite)
with a series of energy states:
for two sets of states with energies $E_1$ and $E_2 > E_1$
and degeneracies (# states at each $E$) $g_1$ and $g_2$
ratio of number of particles in these states is

$$\frac{n(E_2)}{n(E_1)} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/T} \tag{1}$$

where I put $k = 1$, i.e., $kT \rightarrow T$



Example: atomic hydrogen, at $T$
*Q: ratio of ground ($1S$) to 1st excited state ($2P$) populations?*

Atomic hydrogen (H I):

- energy levels: $E_n = -B_{\mathrm{H}}/n^2$ for $n \geq 1$
- angular momenta degeneracies: $g_\ell = 2\ell + 1$

1$S$: $n = 1 \rightarrow E(1S) = -B$; $\ell = 0 \rightarrow g(1S) = 1$

2$P$: $n = 2 \rightarrow E(2P) = -B/4$; $\ell = 1 \rightarrow g(2P) = 3$

$$\frac{n(2P)}{n(1S)} = 3e^{-3B/4T} = 3e^{-120{,}000\,\mathrm{K}/T} \tag{2}$$

*Q: sanity checks–is this physically reasonable?*

*Q: how does this ratio change if plasma is partially ionized i.e., contains both* H I *and* H II$=$ H$^+ = p$*?*

Note: H is bound system $\rightarrow$ discrete energies
we now broaden analysis to include unbound systems
$\rightarrow$ continuous energies, momenta

# Statistical Mechanics in a Nutshell

classically, **phase space** $(\vec{x}, \vec{p})$
completely describes particle state
but quantum mechanics $\rightarrow$ uncertainty $\Delta x \Delta p \geq \hbar/2$
semi-classically: min phase space "volume"
$(dx\ dp_x)(dy\ dp_y)(dz\ dp_z) = h^3 = (2\pi\hbar)^3$
per quantum state of fixed $\vec{p}$

define "occupation number" or "distribution function" $f(\vec{x}, \vec{p})$:
number of particles in each phase space "cell"
*Q: $f$ range for fermions? bosons?*

$$dN = g f(\vec{x}, \vec{p}) \frac{d^3\vec{x}\ d^3\vec{p}}{(2\pi\hbar)^3} \tag{3}$$

where $g$ is # internal (spin/helicity) states:
*Q: $g(e^-)$? $g(\gamma)$? $g(p)$?*

Fermions: $0 \leq f \leq 1$ (Pauli)
Bosons: $f \geq 0$ $g(e^-) = 2s(e^-) + 1 = 2$ electron, same for $p$
$g(\gamma) = 2$ (polarizations) photon

Particle phase space occupation $f$ determines bulk properties

Number density

$$n(\vec{x}) = \frac{d^3N}{d^3x} = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ f(\vec{p},\vec{x}) \tag{4}$$

Mass-energy density

$$\varepsilon(\vec{x}) = \rho(\vec{x})c^2 = \langle En \rangle = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ E(p)\ f(\vec{p},\vec{x}) \tag{5}$$

Pressure

$$P(\vec{x}) = \langle p_i v_i n \rangle_{\mathrm{direction}\,i} = \frac{\langle pvn \rangle}{3} = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ \frac{p\,v(p)}{3}\ f(\vec{p},\vec{x}) \tag{6}$$

*Q: these expressions are general–simplifications in FLRW?*

FRLW universe:

- homogeneous $\longrightarrow$ no $\vec{x}$ dep
- isotropic $\longrightarrow$ only $\vec{p}$ magnitude important $\longrightarrow$ $f(\vec{p}) = f(p)$

in **thermal equilibrium**:

▷ Boson occupation number

$$f_{\mathsf{b}}(p) = \frac{1}{e^{(E-\mu)/kT} - 1} \tag{7}$$

▷ Fermion occupation number

$$f_{\mathsf{f}}(p) = \frac{1}{e^{(E-\mu)/kT} + 1} \tag{8}$$

Note: $\mu$ is "chemical potential" or "Fermi energy"
$\mu = \mu(T)$ but is *independent* of $E$



If $E, \mu \ll T$: both $\longrightarrow$ $f = e^{-(E-\mu)/kT}$
$\longrightarrow$ *Boltzmann distribution*

## Chemical Potential & Number Conservation

For a particle species in thermal equilibrium

$$f(p;T,\mu) = \frac{1}{e^{(E-\mu)/kT} \pm 1} \tag{9}$$

What is $\mu$, and what does it mean physically?

First, what if $\mu = 0$
then $f, n, P$ depend only on $T$
$\rightarrow$ everything at same $T$ has same $\rho, P$!
sometimes true! *Q: examples?* but not always!

but $n$ often conserved
$\rightarrow$ fixed by initial conditions, not $T$
$\rightarrow$ if particle number conserved, $\mu$ determined
by solving $n_{\text{cons}} = n(\mu, T) \rightarrow \mu(n_{\text{cons}}, T)$
so: $\boxed{\mu \neq 0 \Leftrightarrow \text{particle number conservation}}$

if "chemical" equilibrium:

- rxns change particle numbers among species
- equilibrium: forward rate = reverse rate

$a + b + \cdots \leftrightarrow A + B + \cdots$

then

$$\sum_{\text{initial particles} i} \mu_i = \sum_{\text{final particles} f} \mu_f \tag{10}$$

sum of chemical potentials "conserved"

# Equilibrium Thermodynamics

Gas of mass $m$ particles at temp $T$:
$n$, $\rho$, and $P$ in general complicated
because of $E(p) = \sqrt{p^2 + m^2}$
but simplify in ultra-rel and non-rel limits

*Non-Relativistic Species*

$E(p) \simeq m + p^2/2m$, $T \ll m$
for $\mu \ll T$: Maxwell-Boltzmann, same for Boson, Fermions

for non-relativistic particles = *matter*
energy density, number density vs $T$?
*Q: recall* $n(a), \rho(a)$ *and* $T(a)$*?*

## *Non-Rel Species*

number density

$$n = \frac{g}{(2\pi\hbar)^3} e^{-(mc^2-\mu)/kT} \int d^3p \; e^{-p^2/2mkT} \tag{11}$$

$$= g e^{-(mc^2-\mu)/kT} \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \tag{12}$$

energy density:

$$\rho c^2 = \langle En \rangle = \varepsilon_{\text{rest mass}} + \varepsilon_{\text{kinetic}} \tag{13}$$

$$= mc^2 \, n + \frac{3}{2} \, kT \, n \tag{14}$$

$$\simeq \varepsilon_{\text{rest mass}} = mc^2 n \tag{15}$$

pressure

$$P = \frac{\langle pvn \rangle}{3} = \frac{\langle p^2 n/m \rangle}{3} = \frac{2}{3} \varepsilon_{\text{kinetic}} \tag{16}$$

$$= nkT \ll \rho c^2 \tag{17}$$

recover the ideal gas law!

# The Ratio of Baryons to Photons

The number of barons per photon
is the "baryon-to-photon ratio" $\eta \equiv n_B/n_\gamma$

photons not conserved in general:
e.g., Brehmsstrahlung $e \to e + \gamma$
so chem pot $\mu_e = \mu_e + \mu_\gamma \to \mu_\gamma = 0$
$\to n_\gamma \sim T^3$: fixed by $T$ alone

baryons conserved:
#baryons = const in comoving vol
$d(n_B a^3) = 0 \to n_B \propto a^{-3}$
$\to$ so $\mu_B(T) \neq 0$ enforces this scaling

Thus we have

$$\eta = \frac{n_{B,0} a^{-3}}{n_{\gamma,0}(T/T_0)^3} = \left(\frac{T_0}{a\,T}\right)^3 \eta_0 \tag{18}$$

baryon number conservation: $n_{\rm B} \propto a^{-3}$
thermal photons: $n_\gamma \propto T^3$

so as long as $T \sim 1/a$ then
$\eta = const$! baryon-to-photon ratio conserved!
thus we expect $\eta_{\rm BBN} = \eta_{\rm CMB} = \eta_0$!

numerically (from BBN, CMB anisot):

$$\eta_0 \sim 6 \times 10^{-10} \ll 1 \quad (19)$$

*huge* number of photons per baryon! never forget!

but $\rho_B/\rho_\gamma \sim m_B n_B / T n_\gamma \sim \eta m_B / T \neq const$

# Director's Cut Extras

# Kinetic Theory of Pressure due to Particle Motions

consider cubic box, sidelength $L$ (doesn't really need to be cubic)
contain "gas" of $N$ particles: can be massive or massless
particles collide with walls, bounce back elastically
particles exert force on wall $\leftrightarrow$ wall on particles
this lead to bulk *pressure*

focus on one particle, and its component of motion
in one (arbitrary) axis $x$: speed $v_x$, momentum $p_x$

- *elastic* collision: $p_{x,init} = -p_{x,fin} \rightarrow \delta p_x = 2p_x$
- collision time interval for same wall: $\delta t_x = v_x/2L$
- single-particle *momentum transfer* (force) per wall:
  $F_x = \delta p_x / \delta t_x = p_x v_x / L$
- *single-particle force* per wall area:
  $P = F_x/L^2 = p_x v_x / L^3 = p_x v_x / V$

*Q: total pressure?*

total pressure is sum over all particles:

$$P = \sum_{\text{particles } \ell=1}^{N} \frac{p_x^{(\ell)} v_x^{(\ell)}}{V} \tag{20}$$

can rewrite in terms of an average momentum flux

$$P = \frac{N}{V} \frac{\sum_{\ell=1}^{N} p_x^{(\ell)} v_x^{(\ell)}}{N} = \langle p_x v_x \rangle n \tag{21}$$

where $n = N/V$ is *number* density
$\langle p_x \rangle n$ would be average *momentum density* along $x$
and $\langle p_x v_x \rangle n$ is average *momentum flux* along $x$

if particle gas has isotropic momenta, then

$$\langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_x \rangle = \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle = \frac{1}{3} \langle pv \rangle \tag{22}$$

so $\boxed{P = \frac{1}{3} \langle pv \rangle n}$

## *Ultra-Relativistic Species*

$E(p) \simeq cp \gg mc^2$ (i.e., $kT \gg mc^2$):
Also take $\mu = 0$ ($\mu \ll kT$)

energy density, number density?
*Q: recall the answers?*

for relativistic bosons

number density

$$
\begin{aligned}
n_{\text{rel,b}} &= \frac{g}{(2\pi\hbar)^3} \int d^3p \ \frac{1}{e^{cp/kT}-1} \\
&= \frac{4\pi g}{(2\pi\hbar)^3} \int dp \ p^2 \ \frac{1}{e^{cp/kT}-1} = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \int_0^\infty du \ u^2 \ \frac{1}{e^u - 1} \\
&= g\frac{\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \propto T^3
\end{aligned}
$$

where

$$
\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.20206\ldots \qquad (23)
$$

relativistic fermions:

$$n_{\rm rel,f} = \frac{3}{4} n_{\rm rel,b} \tag{24}$$

so $n \propto T^3$ for both

e.g., CMB today: $n_{\gamma,0} = 411\ {\rm cm}^{-3}$

energy density: relativistic bosons

$$\begin{aligned}
\rho_{\rm rel,b} c^2 &= \frac{g}{(2\pi\hbar)^3} \int d^3p\ cp\ \frac{1}{e^{cp/kT} - 1} \\
&= \frac{g}{2\pi^2} \frac{(kT)^4}{(\hbar c)^3} \int_0^\infty du\ u^3\ \frac{1}{e^u - 1} \\
&= g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3}
\end{aligned}$$

and for fermions

$$\rho_{\rm rel,f} = \frac{7}{8} \rho_{\rm rel,b} \tag{25}$$

so $\rho \propto T^4$ for both

pressure

$$P_{\text{rel}} = \left\langle \frac{pv}{3} n \right\rangle = \frac{1}{3} \rho_{\text{rel}} c^2 \tag{26}$$

since $v = c$

$P \propto T^4$

# Temperature Evolution

If in therm eq, maintain photon occ. #

$$f(p) = \frac{1}{e^{p/T} - 1} \tag{27}$$

but $cp = h\nu = hc/\lambda \propto 1/a(t)$:
$\Rightarrow p = p_0/a$

w/o interactions, const # $\gamma$ per mode $p$
$\Rightarrow f(p) = const$
$\Rightarrow p(t)/T(t) = p_0/T_0$
$\Rightarrow \boxed{T/T_0 = p/p_0 = 1/a = 1 + z}$
e.g., at $z = 3$, CMB $T = 4T_0 \simeq 11$ K
(measured in QSO absorption line system!)

recall: used $w = 1/3$ to show $\rho_\gamma \propto a^{-4}$
but blackbody $\rho_\gamma \propto T^4$
together $T \propto 1/a$ (OK!)