Astro 596/496 PC
Lecture 30
April 5, 2010

Announcements:
- PF5 was due at noon
- PS5 out, due in class next **Monday**

Last time: inflation and cosmic lumpiness
classical inflaton field removes all spatial perturbations
$\rightarrow$ has zero point modes, inflaton excitations

Today:
quantum fluctuations in inflaton field
inevitably lead to density perturbations
if this really happened:
we are descendant of quantum fluctuations!

Quantum Mechanics to the Rescue

but quantum mechanics exists and is mandatory
classical ϕ field $\rightarrow$ quantized as inflatons
think $\vec{E}, \vec{B}$ vs photons

inflaton field must contain quantum fluctuations
before, during inflation

What we want: statistical properties of fluctuations
- typical magnitude of fluctuations $\delta\phi$
- how $\delta\phi$ depends on lengthscales
- corresponding fluctuations in ρ_ϕ
- correlations at different length scales

Fluctuation Amplitude: Rough Estimate

quantum fluctuation $\rightarrow$ turn to uncertainty principle

$$\delta E \; \delta t \sim \hbar \sim 1 \tag{1}$$

recall: energy density is

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V \tag{2}$$

if perturbation from classical: $\phi(t,\vec{x}) = \phi_{\rm cl}(t) + \delta\phi(t,\vec{x})$,
then for small $\delta\phi$,

$$\delta\rho \sim (\nabla\delta\phi)^2 + V'(\phi_{\rm cl})\delta\phi \approx (\nabla\delta\phi)^2 \tag{3}$$

since slow roll $\rightarrow$ V' small (flat potential)

Q: what is characteristic volume for fluctuation?
Q: what is characteristic timescale δt?

H^{-1} is only lengthscale in problem
so $\nabla \delta\phi \sim \delta\phi / H^{-1} \Rightarrow \delta\rho \sim H^2 (\delta\phi)^2$
so in Hubble volume $V_H = d_H^3 = H^{-3}$, energy fluctuation is

$$\delta E = \delta\rho \; V_H = \frac{(\delta\phi)^2}{H} \tag{4}$$

characteristic timescale is $\delta t \sim 1/H$, so

$$\delta E \; \delta t \sim \frac{(\delta\phi)^2}{H^2} \sim 1 \tag{5}$$

and typical (root-mean-square) inflaton fluctuation is

$$\boxed{\delta\phi \sim H} \tag{6}$$

had to be! H is the only other
dimensionally correct scale in the problem!

Note: $H \sim const$ during inflation
all fluctuations created with $\sim$ same amplitude

What Just Happened?

To summarize:

- *classically*, inflaton field ϕ_{cl} quickly inflates away any of its initial perturbations
- but *quantum fluctuations* $\delta\phi$ unavoidable created and persist throughout inflation
- in any region, amplitude $\delta\phi(\vec{x})$ random but *typical* value $\delta\phi \sim H$

Q: what do the presence of inflaton fluctucations mean for inflationary dynamics in different regions?

Q: what consequences/signatures of fluctuations might remain after inflation?

Fluctuation Evolution and the Cosmic Horizon

in presence of fluctuations $\delta\phi$ and $\delta\rho_\phi$
can view inflationary universe as ensemble of "sub-universes"
evolving independently–same slow roll, but
with different ϕ, ρ_ϕ at a fixed t
classical discussion $\longrightarrow$ ensemble average
now want behavior typical deviation from mean

particle horizon $\sim H^{-1}$ critical
- already saw: sets scale for fluctuation
- also "shuts off" fluctuation evolution

consider perturbation of lengthscale λ
- leaves horizon when $H \sim 1/\lambda$
- then can't evolve further: keeps same $\delta\rho/\langle\rho\rangle$
- until after inflation, when re-enters horizon

Bottom line: at any given scale λ
relevant perturbation is the one born
during inflation when $\lambda \sim 1/H$

dimensionless perturbation amplitude:
fraction of mean density in horizon $\delta_H \equiv \delta\rho/\langle\rho\rangle$

on scale λ, amplitude fixed at horizon exit
$\delta\phi \sim H$ (in fact, $H/2\pi$)
$\rightarrow$ perturbed universe starts inflating at higher ϕ
or undergoes inflation for different duration $\delta t \simeq \delta\phi/\dot{\phi}$
this gives an additional expansion

$$\delta \ln a = \frac{\delta a}{a} = H\delta t = \frac{H^2}{2\pi\dot{\phi}} \tag{7}$$

but inflation exit is set at fixed ϕ_{end}
and $V_{\text{end}} \sim \rho_{\text{end}}$

perturbed energy density at end of inflation set by different expansion at inflation exit:

$$\delta_{\text{H}} \equiv \frac{\delta\rho}{\langle\rho\rangle} \tag{8}$$

$$\sim \frac{\delta(a^3 V_{\text{end}})}{\langle a^3 V_{\text{end}}\rangle} \sim \frac{\delta a}{a} = \frac{H^2}{2\pi\dot{\phi}} \tag{9}$$

evaluated at any scale λ at horizon crossing
i.e., when $\lambda \sim 1/H$

$\Rightarrow$ *density perturbations created at all lengthscales!*

caution: quick-n-dirty result

but gives right answer
in particular, fluctuation indep of lengthscale

What Just Happened? ...Part Deux

the *classical* behavior of a slow-rolling ϕ
lead to homogeneity, isotropy
regardless of initial conditions
$\Rightarrow$ fixes horizon, flatness, monopole problem

the *quantum* fluctuations in ϕ
lead to density perturbations on all lengthscales
including scales $\gg d_{\text{hor}}$ today
these perturbations form the "seeds" for cosmic structures!

quantum mechanics & uncertainty principle
essential for the existence of cosmic structure

"The Universe is the ultimate free lunch."
– Alan Guth

Evolution of Quantum Perturbations

Write spatial fluctuations in inflaton field
as sum (integral) of Fourier modes:

$$\delta\phi(t,\vec{x}) = \sum_{\vec{k}} \delta\phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}_{\rm com}} \tag{10}$$

where $k = k_{\rm com} = 2\pi/\lambda_{\rm com}$ is comoving wavenumber

classical part of $\delta\phi_{\vec{k}}$ inflated away
but quantum part crucial

in Director's Cut notes:
- inflaton field begins in vacuum state
- evolves as a quantum harmonic oscillator

$\longrightarrow$ dominated by vacuum=ground state

Inflation Spectrum
Slightly Tilted Scale Invariance

recall: perturbation leaving horizon have very similar amplitude during inflation $\rightarrow$ nearly same for all lengthscales $\leftrightarrow k$

perturbation rms amplitude

$$\delta^2_{\rm inf}(k) \propto k^{-6\epsilon+2\eta} \tag{11}$$

$\star$ successful inflation $\Leftrightarrow$ slow roll $\Leftrightarrow \epsilon, \eta \ll 1$ demands
perturbation spectrum nearly independent of scale
nearly "self-similar," without characteristic scale
"Peebles-Harrison-Zel'dovich" spectrum

$\star$ successful inflation must end $\rightarrow \epsilon, \eta \neq 0$
demands small departures from scale-invariance
"tilted spectrum"

Inflation Spectrum
Statistical Properties

⋆ Recall: inflaton quantum modes $\leftrightarrow$ harmonic oscillator
dominated by vacuum $\leftrightarrow$ ground state $\|\psi_{\rm sho}(x)\|^2 \sim e^{-x^2/2\Delta x^2}$
$\phi_k \leftrightarrow x$ fluctuations are statistically Gaussian
i.e., perturbations of all sizes occur, but
probability of finding perturbation of size $\delta(R)$
on scale R is distributed as a Gaussian

⋆ inflaton perturbations $\rightarrow$ reheating
$\rightarrow$ radiation, matter perturbations
same levels in both: "adiabatic"

⋆ so far: only looked at density (scalar) perturbations
but also tensor perturbations $\rightarrow$ gravity waves!

⋆⋆⋆⋆⋆ All of these are bona fide *pre*dictions of inflation
and are testable! *Q: how?*

Testing Inflation: Status to Date

test by measuring density fluctuations
and their statistical properties
on various scales at various epochs

CMB at large angles (large scales, decoupling)

- nearly scale invariant! woo hoo! (COBE 93)
- Gaussian distribution (COBE, WMAP)
  `www: 3-yr WMAP` T `distribution`
  or nearly so...see Yadav & Wandelt (2007)
- WMAP 2008: evidence for tilt! favors large scales ("red")!
  $\alpha = -0.040^{+0.014}_{-0.013}$ nonzero at $\sim 3\sigma$!

These did not have to be true!
Not guaranteed to be due to inflation
but very encouraging nonetheless

Inflation Scorecard

Summary:
Inflation designed to solve horizon, flatness, smoothness
does this, via accelerated expansion driven by inflaton

But unexpected bonus: structure
inflaton field has quantum fluctuations
imprinted before horizon crossing
later return as density fluctuations
$\rightarrow$ inflationary seeds of cosmic structure?!

Thus far: observed cosmic density fields
have spectrum, statistics as predicted by inflation

Future:
- gravity wave background
  (directly or by CMB polarization)
- departures from scale invariance $\rightarrow$ probes of $V(\phi)$?

Director's Cut Extras

Fluctuation Spectrum: In More Detail

Starting point of more rigorous treatment
in equation of motion

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi + V'(\phi) = 0 \quad (12)$$

write field as sum

$$\phi = \phi_{\text{classical}}(t) + \delta\phi(t, \vec{x}) \quad (13)$$

- **classical amplitude** $\phi_{\text{cl}}(t)$
  spatially homogeneous: *smooth, classical, background* field
  evolves according to classical equation of motion
  $\rightarrow$ this has been our focus thus far; now add
- **quantum fluctuations** $\delta\phi(t, \vec{x})$
  these can vary across space and with time

decompose spatial part of fluctuations into plane waves

$$\delta\phi(t,\vec{x}) = \sum_{\vec{k}} \delta\phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}_{\text{com}}} \tag{14}$$

convenient to label Fourier modes by

comoving wavelength $\lambda \equiv \lambda_{\text{com}}$, wavenumber $k \equiv k_{\text{com}} = 2\pi/\lambda_{\text{com}}$

but *physical* wavelength $\lambda_{\text{phys}} = a\lambda_{\text{com}}$, wavenumber $k_{\text{phys}} = k/a$

as long as quantum perturbations $\delta\phi$ small (linear evolution)

each wavelength–i.e., scale–evolves independently

$\rightarrow$ main reason to use Fourier modes

classically $\delta\phi = (\delta\phi)^2 = 0$ by definition!

Q: what is physical significance of quantum excitations in ϕ?

The Quantum Inflaton Field

quantum mechanically:

- true ϕ has fluctuations around background value
- each $\vec{k}$ mode $\leftrightarrow$ independent quantum states (oscillators)
- mode fluctuations *quantized* $\rightarrow$ quanta are inflaton particles
  analogous to photons as EM quanta
- occupation numbers: $n_{\vec{k}} > 0 \rightarrow$ real particles present
- if $n_{\vec{k}} = 0 \rightarrow \langle \delta\phi \rangle = 0$ no particles (vacuum/ground state)
  but zero-point fluctuations still present $\langle \delta\phi^2 \rangle \neq 0$

Fluctuation Lagrangian

expand each $\vec{k}$ mode around classical value

$$\mathcal{L}_{\vec{k}} = \frac{1}{2}\delta\dot{\phi}_{\vec{k}}^2 - \frac{1}{2}\frac{k^2}{a^2}\delta\phi_{\vec{k}}^2 - \frac{1}{2}V''(\phi_{\text{cl}})\delta\phi_{\vec{k}}^2 - V(\phi_{\text{cl}}) \quad (15)$$

$$\approx \frac{1}{2}\delta\dot{\phi}_{\vec{k}}^2 - \frac{1}{2}\frac{k^2}{a^2}\delta\phi_{\vec{k}}^2 \quad (16)$$

where slow roll $\rightarrow$ potential terms small
$\rightarrow$ a **massless simple harmonic oscillator**

$\delta\phi$ vacuum state: zero point fluctuations
formally same a quantum harmonic oscillator!
for *each k mode*, fluctuation *amplitudes random*
but probability distribution is like $n = 0$ oscillator

$$P(\delta\phi_{\vec{k}}) \propto e^{-\delta\phi_{\vec{k}}^2/2\sigma_{\vec{k}}^2} \quad (17)$$

where variance $\sigma_{\vec{k}}^2 = \langle\delta\phi_{\vec{k}}^2\rangle$
$\rightarrow$ vacuum fluctuation amplitudes have *gaussian* distribution

Total ϕ energy density is $\rho_\phi = \rho_{\text{background}} + \rho_{\text{zeropoint}} + \rho_{\text{particles}}$
pre-inflation: could have $\rho_{\text{particles}} \neq 0$
in fact: if thermalized, $\rho_{\text{particles}} \propto T^4$ (radiation)
$\rightarrow$ inflation only begins when $\rho_{\text{background}} \gg \rho_{\text{particles}}$
Q: what happens to inflatons after inflation begins?

after inflation begins, universe rapidly expanded, cooled
inflatons diluted away
$\longrightarrow$ inflation field driven to **vacuum (ground) state**

Since ϕ in quantum vacuum state: fluctuations are zero-point
$\longrightarrow$ **gaussian distribution** of amplitudes in each k mode
$\longrightarrow$ strong prediction of slow-roll inflation

now want to solve for size of rms σ_k at each mode

classically, perturbations have equation of motion

$$\frac{d^2}{dt^2}\delta\phi + 3H\frac{d}{dt}\delta\phi + \frac{k^2}{a^2}\delta\phi + V''\delta\phi = 0 \quad (18)$$

$$\frac{d^2}{dt^2}\delta\phi + 3H\frac{d}{dt}\delta\phi + \frac{k^2}{a^2}\delta\phi \approx 0 \quad (19)$$

(in slow roll: V'' term negligible)

Sketch of Quantum Treatment

Promote $\delta\phi \rightarrow$ operator $\widehat{\delta\phi}$
plane wave expansion: $\widehat{\delta\phi} = \sum_{\vec{k}} \widehat{\delta\phi}_{\vec{k}}$
introduce annihilation, creation operators $\hat{a}_{\vec{k}}$, $\hat{a}^{\dagger}_{-\vec{k}}$, then

$$\widehat{\delta\phi}_{\vec{k}} = w_k(t)\, \hat{a}_{\vec{k}} + w_k^*(t)\, \hat{a}^{\dagger}_{-\vec{k}} \tag{20}$$

where $w_k(t)$ is a solution of field equation

$$\ddot{w}_k + 3H\dot{w}_k + \left(\frac{k}{a}\right)^2 w_k = 0 \tag{21}$$

Compare limits:

- $k/a \gg H \rightarrow k \gg aH \rightarrow \lambda \ll 2\pi d_{H,\mathrm{com}}$
  Q: physical interpretation of limit?
  w_k evolves as harmonic oscillator (free massless field)
- $k/a \ll H \rightarrow k \ll aH \rightarrow \lambda \gg 2\pi d_{H,\mathrm{com}}$
  Q: physical interpretation of limit?
  $\dot{w}_k \propto a^{-3} \rightarrow 0 \rightarrow w_k$ value "frozen"

Inflation Perturbations: Evolution and Horizons

sub-horizon scales $\lambda \ll 2\pi d_{H,\rm com}$
inflaton fluctuations $\delta\phi$ are causally connected
evolve like harmonic oscillator $\rightarrow$ rms amplitude $\langle |w_k|^2 \rangle$ *constant*

but cosmic acceleration during inflation $\rightarrow$ $d_{H,\rm com}$ *shrinks*
since $\dot{d}_{H,\rm com} = d(aH)^{-1}/dt = d(\dot{a}^{-1})/dt = -\ddot{a}/\dot{a}^2 < 0$ during inf
$d_{H,\rm com}$ shrinkage: initially sub-horizon scales $\rightarrow$ super-horizon

super-horizon scales $\lambda \gg 2\pi d_{H,\rm com}$
fluctuations out of causal contact
amplitude "frozen in" until post-inflation $d_{H,\rm com}$ regrows

Inflation Perturbations: Spectrum of Amplitudes

examine fluctuations from vacuum
$\longrightarrow$ find expected amplitudes w_k

since fluctuations have *quantum* origin
- cannot predict definite values for mode amplitudes, phases
- but *can* predict statistical properties

for *different* modes $\vec{k}$ and $\vec{k}'$,
Q: what do we expect?

for *different* modes $\vec{k}$ and $\vec{k}'$,
expectation is

$$\langle \widehat{\delta\phi}_{\vec{k}} \widehat{\delta\phi}_{\vec{k}'} \rangle = w_k w_{k'} \langle \hat{a}_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}'} \rangle + \text{c.c.} = 0 \tag{22}$$

because $\langle \hat{a}_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}'} \rangle = \langle \hat{a}_{\vec{k}} \rangle \langle \hat{a}^{\dagger}_{\vec{k}'} \rangle = 0$
$\Rightarrow$ modes are statistically independent
note: true even if $|\vec{k}| = |\vec{k}'| = k$ but $\vec{k} \cdot \vec{k}' = 0$
i.e., different directions $\vec{k} = k\hat{x}, \vec{k}' = k\hat{y}$
$\Rightarrow$ phase $e^{i\vec{k}\cdot\vec{x}}$ is random

for a single mode k, vacuum expectation is

$$\langle \widehat{\delta\phi}^2_{\vec{k}} \rangle = |w_k|^2 \langle \hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} \rangle = |w_k|^2 \neq 0 \tag{23}$$

$$= \frac{H^2}{2L^3k^3} \tag{24}$$

where last expression
- from full quantum calculation, in box of size L
- to be evaluated at horizon crossing: $k_{\text{phys}} = H \rightarrow k = aH$

each in phase space volume

$$d^3x d^3k = \frac{1}{(2\pi L)^3} 4\pi k^2 dk = \frac{4\pi k^3}{(2\pi L)^3} \frac{dk}{k} \tag{25}$$

then fluctuation amplitude is

$$P_\phi(k) \frac{dk}{k} \equiv \frac{4\pi k^3}{(2\pi L)^3} |\delta\phi_k|^2 \frac{dk}{k} = \left(\frac{H}{2\pi}\right)^2 \frac{dk}{k} \tag{26}$$

and so the phase space fluctuation density in ϕ is

$$P_\phi(k) = \left(\frac{H}{2\pi}\right)^2_{k=aH} \tag{27}$$

as before, but now

- explicitly seen independence of k
- know when to evaluate: at horizon crossing $k = aH$

Fluctuation Evolution and Spectrum

consider some fixed (comoving) scale $\lambda = 2\pi/k$ key idea: causal physics acts until $\lambda > d_{\mathrm{H,com}}$: "horizon crossing"
$\rightarrow$ quantum fluctuations laid down while inside $d_{\mathrm{H,com}}$
"frozen in" once outside of $d_{\mathrm{H,com}}$

from last time: quantum analysis gives fluctuation variance

$$\left\langle (\delta\phi_k)^2 \right\rangle = \left(\frac{H}{2\pi}\right)^2_{k=aH} \tag{28}$$

to be evaluated at horizon crossing: $k = 1/d_{\mathrm{H,com}} = aH$

Fluctuation Evolution and Spectrum

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Inflationary Density Perturbations: Spectrum

Recall: density fluctuations $\rightarrow$ start inflating earlier (later)
$\rightarrow$ more (less) expansion than average
extra scale factor boost $\delta a/a = H\delta t = H\delta\phi/\dot{\phi}$ $\rightarrow$ larger volume
$\rightarrow$ density perturbations have mean square

$$\delta^2_{\mathrm{inf}}(k) \equiv \left(\frac{\delta\rho}{\rho}\right)^2_k \quad (30)$$

$$\sim \left(\frac{\delta a}{a}\right)^2 = \left(\frac{H}{\dot{\phi}}\right)^2 (\delta\phi)^2 = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \quad (31)$$

evaluated at $aH = k$

slow roll: H, $\dot{\phi}$ slowly varying
$\rightarrow$ expect fluctuation amplitude $\sim H^4/\dot{\phi}^2 \sim const$
over wide range of k

In particular: slow roll $\dot\phi = -3V'/H$,
and $H^2 = V/3m_{\rm pl}^2$, which gives

$$\delta^2_{\rm inf}(k) = \frac{1}{12\pi^2 m_{\rm pl}^6}\left(\frac{V^3}{V'^2}\right) = \frac{1}{24\pi^2 m_{\rm pl}^4}\left(\frac{V}{\epsilon}\right) \qquad (32)$$

where $\epsilon = m_{\rm pl}(V'/V)^2/2$

anticipating $\sim$ power law behavior,
define $\delta^2_{\rm inf}(k) \sim k^{\alpha(k)}$
then scale dependence is

$$\alpha(k) = \frac{d\ln\delta^2_{\rm inf}(k)}{d\ln k} \qquad (33)$$

evaluated when comoving scale $k = aH$ crosses horizon
i.e., this relates k to homogeneous a, ϕ values

Underlying physical question:
how do cosmic properties–e.g., $H, \rho \approx V$–*change*
while the universe inflates as it slowly rolls?

- if no change $\rightarrow \dot{\phi} = 0 \rightarrow$ same V, H always $\rightarrow \epsilon = 0$
  all scales see same U when leaving horizon $k = aH$
  $\rightarrow$ all scales have same quantum fluctuations
- but *slow* roll $\neq$ *no* roll!
  $\dot{\phi} \neq 0 \rightarrow$ U properties *do* change

recall: $\delta^2_{\rm inf}(k) \propto V/\epsilon$
and as slowly roll $\rightarrow V$ decreasing, ϵ increasing
and horizon scale $d_{H,\rm com}$ also decreases
Q: so which scales get larger perturbations? smaller?

because V decreasing, ϵ increasing

$\delta^2_{\rm inf}(k) \propto V/\epsilon$ decreases with time

$\rightarrow$ smaller perturbations later in slow roll

since horizon scale $d_{H,\rm com}$ decreases

later times $\leftrightarrow$ smaller scales

$\Rightarrow$ slow roll $\rightarrow$ *smaller* perturbations on *smaller* scales

$\Rightarrow$ perturbation spectrum *tilted* to large scales $\rightarrow$ small k

in slow roll, $k = aH$ change mostly due to a:

$$d \ln k \approx d \ln a = \frac{da}{a} = H \ dt \tag{34}$$

recast in terms of inflaton potential

$$= \frac{H d\phi}{\dot{\phi}} = -3 \frac{H^2}{V'} d\phi \tag{35}$$

and so

$$\frac{d}{d \ln k} = -m_{\rm pl}^2 \frac{V'}{V} \frac{d}{d\phi} \tag{36}$$

Using this, can show:

$$\alpha(k) = \frac{d \ln \delta_{\rm inf}^2(k)}{d \ln k} = -6\epsilon + 2\eta \tag{37}$$

i.e., perturbation spectrum $\delta_{\rm inf}^2(k) \propto k^{-6\epsilon+2\eta}$

Major result!

Q: why? what does this mean physically? for cosmology? for inflation?