Astro 596/496 PC
Lecture 32
April 12, 2010

Announcements:

- PS5 due today
- PF6 out – last preflight! – due next Monday noon
- **ICES** availalbe online – please do it!
- Physics Colloquium this week: Peter Axel Lecture
  Michael Ramsey-Musolf, Wisconsin
  "Nuclear Science and the New Standard Model"
  ⇒ cosmic baryogenesis and the LHC

Last time: opened our eyes to the *inhomogeneous* universe
began structure formation
Q: why is this a fundamentally statistical problem?
Q: what is cosmological "bias"?
Q: what measures of structure would be more fair and balanced?
Q: how to quantify cosmic structure?

key measure of cosmic structure: **density contrast**

$$\delta(\vec{x}) = \frac{\delta\rho}{\rho} \equiv \frac{\rho(\vec{x}) - \langle\rho\rangle}{\langle\rho\rangle} \in (-1, \infty) \tag{1}$$

$$\delta_{\vec{k}} = \frac{1}{V} \int \delta(\vec{x}) \; e^{i\vec{k}\cdot\vec{x}} \; d^3\vec{x} \tag{2}$$

where average is over large volume V

Q: what is the order-of-magnitude of the density contrast in this room? of the Galactic ISM?

by definition: $\langle\delta\rangle = \frac{1}{V}\int d^3x\, \delta(\vec{x}) = 0$

would like to study structures on different cosmic lengthscales λ

Q: how to do this using density contrast?

Spectrum of Density Fluctuations

In (large) volume V write $\delta(\vec{x})$ as Fourier series

$$\delta(\vec{x}) = \sum_{\vec{k}} \delta_{\vec{k}}\; e^{-i\vec{k}\cdot\vec{x}} \rightarrow \frac{V}{(2\pi)^3} \int \delta_{\vec{k}}\; e^{-i\vec{k}\cdot\vec{x}}\; d^3\vec{k} \qquad (3)$$

(last expression is continuum limit as $V \rightarrow \infty$)
where Fourier coefficients are

$$\delta_{\vec{k}} = \frac{1}{V} \int \delta(\vec{x}) e^{i\vec{k}\cdot\vec{x}}\; d^3\vec{x} \qquad (4)$$

reality: $\delta(\vec{x})^* = \delta(\vec{x}) \rightarrow \delta^*_{\vec{k}} = \delta_{-\vec{k}}$

Beware!
conventions differ on factors of V, sign of exponential
$\rightarrow$ affects dimensions of δ_k

Fourier mode described by amplitude $|\delta_k|$
and **comoving wavenumbers** $k \equiv k_{\text{comov}} = 2\pi/\lambda_{\text{comov}}$
and $\vec{x}$ is comoving as well
physical values are $d\vec{x}_{\text{phys}} = a(t)d\vec{x}$, $\vec{k}_{\text{phys}} = \vec{k}/a(t)$

Q: what is $\delta_{\vec{k}=0}$*?*
Q: what is connection between $\delta_{\vec{k}}$, $\delta_{\vec{k}'}$ *if* $|\vec{k}| = |\vec{k}'| = k$*?*

Q: how compute a typical value of $\delta\rho/\rho$*?*
what is it for scale k*?*

Fun Fourier Facts

$$\delta_{\vec{k}=0} = \int d^3\vec{x}\ \delta(\vec{x}) = \langle\delta\rangle = 0 \tag{5}$$

by definition!

but deeper reason: small $k \leftrightarrow$ large λ

$k \to 0$ is $\lambda \to \infty$ = whole universe

on largest scales, U better be homogeneous!

so $\delta_{\text{small}\,k} \to 0$

For $|\vec{k}| = |\vec{k}'| = k$, i.e., same mag, different direction

must find same amplitude fluctuations

...else have a preferred direction*

so cosmo principle $\to$ $\delta_{\vec{k}} = \delta_k$

i.e., wavelength is all that counts k magnitude

* In fact, would $\vec{k}$ anisotropy would manifest not as preferred direction in structure distribution in real space, but rather as preferred *orientation* of structures! (thanks to Z. Lukic for pointing this out)

The Power Spectrum

Want a measure of “typical” fluctuation size”

$\langle \delta\rho/\rho \rangle = \langle \delta \rangle = 0$ by definition, but $\langle (\delta\rho/\rho)^2 \rangle = \langle \delta^2 \rangle \neq 0$

$$\left(\frac{\delta\rho}{\rho}\right)^2 = \int d^3\vec{x}\ \delta(\vec{x})^2 \tag{6}$$

$$= \frac{V^2}{(2\pi)^6} \int d^3\vec{x}\ d^3\vec{k}\ d^3\vec{q}\ \ \delta_{\vec{k}}\delta_{\vec{q}}\ e^{-i(\vec{k}+\vec{q})\cdot\vec{x}} \tag{7}$$

$$= \frac{V}{(2\pi)^3} \int d^3\vec{k}\ d^3\vec{q}\ \ \delta_{\vec{k}}\delta_{\vec{q}}\ \delta_{\mathrm{Dirac}}(\vec{k}+\vec{q}) \tag{8}$$

$$= \frac{V}{(2\pi)^3} \int d^3\vec{k}\ \ \delta_{\vec{k}}\delta_{-\vec{k}} \tag{9}$$

$$= \frac{V}{(2\pi)^3} \int d^3\vec{k}\ |\delta_{\vec{k}}|^2 = \frac{V}{(2\pi)^3} \int d^3\vec{k}\ P(k) \tag{10}$$

where $P(k) = |\delta_k|^2$ is the power spectrum

Rewrite in terms of fluctuations per log interval
in wavenumber dk/k:

$$\left(\frac{\delta\rho}{\rho}\right)^2 = \frac{V}{(2\pi)^3}\int d^3\vec{k}\; P(k) = \frac{4\pi V}{(2\pi)^3}\int dk\; k^2\; P(k) \tag{11}$$

$$= \int \frac{4\pi k^3 P(k) V}{(2\pi)^3}\frac{dk}{k} \tag{12}$$

$$\equiv \int \frac{dk}{k}\left(\frac{\delta\rho}{\rho}\right)_k^2 \tag{13}$$

where the variance over $dk/k = d\ln k \sim 1$ is

$$\left(\frac{\delta\rho}{\rho}\right)_k^2 \approx \Delta^2(k) \equiv 4\pi k^3 P(k)\frac{V}{(2\pi)^3} \tag{14}$$

dimensionless measure of fluctuations on scale k

power spectrum $P(k) \Leftrightarrow \Delta^2(k)$
central object in structure formation
`www: Observed power spectrum` *Q: what stands out?*

Observed Power Spectrum

Gross features of $P(k)$:

★ fairly simple shape: roughly, broken power law
roughly, $P(k) \sim k^1$ at low k,
then steepening negative slope, approaching k^{-3}
we will want to understand why

★ break at peak: $k_{\rm peak} \sim 0.02\ h^{-1}{\rm Mpc}^{-1}$
$\rightarrow$ characteristic scale $\lambda_{\rm peak} = 2\pi/k_{\rm peak} \sim 300$ Mpc comoving
we will want to understand what sets this scale!

Features of $\Delta(k) = \sqrt{\Delta^2(k)}$:

★ $\Delta \gtrsim 1$ at $k \gtrsim 0.03$ Mpc $\rightarrow \lambda \lesssim 20$ Mpc
characteristic scale of nonlinearities today

★ $\Delta \ll 1$ at small k: U $\rightarrow$ homogeneous on large scales
cosmo principle vindicated! Good guess, Al!

Enough already with definitions
and lists of observations!

This is cosmology, not stamp collecting!

Now tell me how to understand it all!