Astro 596/496 PC

Lecture 33

April 14, 2010

Announcements:

- PF6 – last preflight! – due next Monday noon
- **ICES** available online – please do it!
- Physics Colloquium this week: Peter Axel Lecture
  Michael Ramsey-Musolf, Wisconsin
  "Nuclear Science and the New Standard Model"
  ⇒ cosmic baryogenesis and the LHC

Last time: density fluctuations as a function of scale

$\longrightarrow$ have to learn to translate: real space $\leftrightarrow$ Fourier space

- Fourier mode amplitudes $\delta_k \rightarrow$ **power spectrum** $P(k) = |\delta_k|^2$
- observed power spectrum: small k, large scales have $P(k) \sim k^1$
  large k, small scales have $P(k) \rightarrow k^{-3}$
- power spectrum peak at $k_{\rm peak} \simeq 0.02\ h^{-1}$ Mpc, $\lambda_{\rm peak} \simeq 300$ Mpc
- fluctuation amplitude on comoving scale $\lambda = 2\pi/k$:
  $$\Delta^2(k) = \frac{V}{2\pi^2} k^3 P(k)$$
- observed $\Delta \gtrsim 1$ at $\lambda \lesssim 20$ Mpc
  in today's extras: variance in $R = 8\,h^{-1}$Mpc sphere:
  $\sigma_8 \simeq \Delta(k = 1/R) = 0.8$: δ large/nonlinear on scales $\lesssim 10$ Mpc

These results cry out for theoretical understanding!

Theory of Cosmological Perturbations

Treat structure formation as initial value problem

- given *initial conditions*: "seeds"
  i.e., adopt spectrum of primordial density perturbations
  prescription for initial $\rho_i(\vec{x})$, $i \in$ baryons, radiation, DM, DE...
  e.g., inflation: scale invariant, gaussian, adiabatic
- follow *time evolution* of $\rho_i(\vec{x})$–i.e., δ_i for each species i
- compare with observed measures of structure

⋆ agreement (or lack thereof) constrains primordial seeds
  e.g., dark matter, inflation, quantum gravity, ...

We want to describe dynamics of cosmic inhomogeneities
Q: which forces relevant? which irrelevant? which scary?

Dynamics Cosmological Perturbations: Overview

Forces/interactions in perturbed, inhomogeneous universe
involve same cosmic particle/field content
as smooth/unperturbed universe
but: can manifest in new/different ways due to spatial variations

Definitely relevant forces on perturbations

- *gravity*: overdensities have extra attraction
  over that of "background" FRW universe
- *pressure*: baryons have thermal pressure $P = nkT$
  photons exert radiation pressure on baryons pre-decoupling
  pressure *gradients* present, unlike in homog. background

Probably irrelevant forces on perturbations (will ignore)

- neutrino interactions with self, other species
- dark matter non-gravity interactions with self, or other species

Scary forces on perturbations (will ignore for now, but worry about)

- if dark energy is a field ϕ, perturbations $\delta\phi$ exert inhomogeneous *negative* pressure
  why scary? unknown underlying physics
- magnetic fields $\rightarrow$ pressure, MHD forces
  why scary? unknown initial conditions (primordial B?)

At minimum: we will want to describe baryons & dark matter as inflationary perturbations grow thru radiation, matter eras
$\longrightarrow$ *gravity* and photon, baryon *pressure* mandatory
schematically:

$$\text{acceleration} = -\text{gravity} + \text{pressure} \tag{1}$$

Q: implications for baryons vs dark matter?

For the species and forces we choose to follow:
Q: how can these be described exactly? approximately?
Q: what formalism needed?

Dynamics of Cosmological Perturbations: Toolbox

need dynamics of inhomogeneous **“fluids”**
in expanding FLRW background
⋆ full treatment: general relativistic perturbation theory
mandatory for some results *Q: which?*
⋆ good-enough treatment: Newtonian dynamics is FLRW
as usual, benefits: intuition & simplicity
costs: limited range of validity

Newtonian Fluid Dynamics & Self-Gravity

Each cosmic species is "fluid" described by fields

- mass density $\rho(\vec{x},t)$
- velocity $\vec{v}(\vec{x},t)$
- pressure $P(\vec{x},t)$: from equation of state $P = P(\rho, T)$

In Newtonian limit: dynamics governed by

mass conservation (continuity) $\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0$

Euler: "$F = ma$" $\rho d\vec{v}/dt = \rho \partial_t \vec{v} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \rho \nabla \Phi$

Note: fixed/non-comoving coords need "*convective derivative*"

$d\vec{v}(\vec{x},t)/dt = (\partial_t + \dot{x}_i \partial_i)\vec{v} = \partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v}$

Newtonian gravity: inverse square → Poisson $\nabla^2 \Phi = 4\pi G \rho$

These are general (albeit Newtonian only)

→ now apply to the Universe

Linear Theory 0: Newtonian, Non-expanding

consider *static*, uniform (infinite) distribution of matter
and introduce small perturbations

$$\rho(\vec{x}) = \rho_0 \; [1 + \delta(\vec{x})] \quad (2)$$

$$v(\vec{x}) = \vec{u}(\vec{x}) \quad (3)$$

$$\Phi_{\mathrm{grav}}(\vec{x}) = \Phi_0 + \Phi_1(\vec{x}) \quad (4)$$

where $\delta \ll 1$, and $\Phi_1, \vec{u}$ "small"

we want: time development of (initially) small perturbations
following Sir James Jeans
many key ideas of full expanding-Universe GR result
already appear here!

Newtonian fluid equations: continuity (mass conservation)

∞

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \quad (5)$$

$$\rho_0 \dot{\delta} + \rho_0 \nabla \cdot \vec{u} \approx 0 \quad (6)$$

Euler (“$F = ma$”);

$$\rho d\vec{v}/dt = \rho \partial_t \vec{v} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p - \rho \nabla \Phi \tag{7}$$

$$\rho_0 \dot{\vec{u}} \approx -\rho_0 c_s^2 \nabla \delta - \rho_0 \nabla \Phi_1 \tag{8}$$

where **adiabatic sound speed** $c_s^2 = \partial p / \partial \rho$

Gravity: Poisson (Gauss’ law = inverse square force)

$$\nabla^2 \Phi = 4\pi G \rho \tag{9}$$

$$\nabla^2 \Phi_1 \approx 4\pi G \rho_0 \delta \tag{10}$$

note inconsistency=cheat! $\nabla^2 \Phi_0 \neq 4\pi G \rho_0$: “Jeans swindle”

can combine to single eq for linearized density contrast:

$$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta \tag{11}$$

Q: behavior for pressureless fluid? “switched-off” gravity?
physical significance? important scales?

Density contrast evolves as

$$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta \tag{12}$$

solutions are of the form

$$\delta(t, \vec{x}) = A e^{i(\omega t - \vec{k} \cdot \vec{x})} \equiv D(t)\ \delta_0(\vec{x}) \tag{13}$$

where $\delta_0(\vec{x}) = e^{-i\vec{k}\cdot\vec{x}}$ is init Fourier amp
and time evolution is set by exponent $\omega(k)$:

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \equiv c_s^2 (k^2 - k_J^2) = \left(\frac{c_s}{k_J}\right)^2 \left[\left(\frac{\lambda_J}{\lambda}\right)^2 - 1\right] \tag{14}$$

key scale: Jeans length

$$k_J = \frac{\sqrt{4\pi G \rho_0}}{c_s} \qquad \lambda_J = \frac{c_s}{\sqrt{G\rho_0/\pi}} \sim c_s \tau_{\rm freefall} \tag{15}$$

associate Jeans mass: $M(\lambda_J/2) = 4\pi/3\ \rho_0 (\pi/k_J)^3 \sim c_s^3/G^{3/2} \rho_0^{1/2}$
Q: physically, what expect for $\lambda < \lambda_J$? $\lambda > \lambda_J$?

perturbation growth $\delta_k(t) = \delta_k(t_0) e^{i\omega t}$, with

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \equiv c_s^2 (k^2 - k_J^2) \tag{16}$$

Jeans length $\sim c_s \tau_{\text{freefall}}$: sound travel distance in freefall time
$\rightarrow \lambda/\lambda_J \sim$ number of pressure wave crossings during freefall

if $k > k_J$ so $\lambda < \lambda_J$, small scales: pressure can repel contraction
ω real: oscillations about hydrostatic equlilb

if $k < k_J$ so $\lambda > \lambda_J$, large scales: pressure ineffective
ω imaginary, exponential collapse
runaway perturbation growth $D(t) = e^{\omega t} \sim e^{+t/t_{\text{freefall}}}$
(also an uninteresting decaying mode $e^{-\omega t}$)

Q: but what about expanding Universe?

Summary of Jeans Analysis in Static Background

Sir James Jeans: Newtonian evolution of density perturbations of non-expanding, static homogeneous background fluid

key Jeans results:

- wavelike solutions, e.g., $\delta_{\vec{k}} \propto e^{i(\vec{k}\cdot\vec{x}-\omega t)} \equiv D(t)\, \delta_0(\vec{x})$
- dispersion relation $\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \approx c_s^2 k^2 - 1/\tau_{\rm frefall}^2$
- critical scale: Jeans wavenumber $k_J = \sqrt{4\pi G \rho_0}/c_s$, Jeans wavelength $\lambda_J = 2\pi/k_J = c_s\sqrt{\pi/G\rho_0} \approx c_s \tau_{\rm freefall}$
- on large scales $k < k_J$: $\omega^2 < 0 \rightarrow D(t) = e^{+\omega t}$ *density contrast grows exponentially for large-scale modes!*

★ **Jeans instability** or **gravitational instability**

Q: physical explanation? why only large-scale modes unstable?

Q: in expanding U, should grav instability be stronger or weaker?

Director's Cut Extras

Correlation Function

Taking $\langle\delta(\vec{x})^2\rangle$ gives $(\delta\rho/\rho)^2_{\mathrm{rms}}$

$\longrightarrow$ overlap of density contrast with itself

(at same point in space)

What about $\xi(\vec{r}) = \langle\delta(\vec{x})\delta(\vec{x}+\vec{r})\rangle$ (fixed $\vec{r}$, avg over $\vec{x}$)

(two-point or auto-) correlation function

- physical significance?

  what if ρ at each space point independent of all other points?

  opposite case: what if strictly periodic (lattice)?
- sign(s)? meaning of sign(s)?
- behavior at large, small $|\vec{r}|$?
- significance of r at which $\xi(r) = 0$?
- dependence on $\hat{r} = \vec{r}/|\vec{r}|$?

Correlation function: avg of density contrast overlap with itself, “lagged” by spacing $\vec{r}$:

$$\xi(\vec{r}) = \langle\delta(\vec{x})(\vec{x}+\vec{r})\rangle = \frac{1}{V}\int \delta(\vec{x})\ \delta(\vec{x}+\vec{r})\ d^3\vec{x} \tag{17}$$

- physically: given δ somewhere, measures typical δ separated by $\vec{r}$
- if each space point independent of all others, no matter how close, then:
  $\xi(\vec{r}) = 0$ for $\vec{r} \neq 0$
- but even if this were ever true, local physics *must* remove independence
- since $\delta \in (-1,\infty)$, ξ can be negative (must be for some r!)

Demo–toy model transparencies

Q: if structure in a lattice, what does ξ measure?

Q: what is significance of first zero of ξ?

Correlation function in an idealized "Lattice Universe"

- if lattice of galaxy clusters, ξ oscillates with lattice periodicity
  $\rightarrow$ gives typical cluster size, and typical cluster separation
  true even if not lattice

Correlation function generally:

- first $\xi(\vec{r}) = 0$ gives typical cluster size
- small $\vec{r}$: must have $\xi \rightarrow (\delta\rho/\rho)^2 > 0$
  large $\vec{r}$: correlations must vanish $\xi \rightarrow 0$
  (cosmo principle/horizons)
- isotropy: $\xi(\vec{r}) = \xi(r)$ independent of direction

In Fourier space:

$$\xi(\vec{r}) = \frac{1}{V}\int \delta(\vec{x})\ \delta(\vec{x}+\vec{r})\ d^3\vec{x} \tag{18}$$

$$= \frac{V}{(2\pi)^6}\int \delta_{\vec{k}}\,\delta_{\vec{q}}\,e^{-i(\vec{k}+\vec{q})\cdot\vec{r}}e^{-i\vec{q}\cdot\vec{r}}\quad d^3\vec{k}\ d^3\vec{k}\quad d^3\vec{x} \tag{19}$$

$$= \frac{V}{(2\pi)^3}\int \delta_{\vec{k}}\,P(k)\ e^{-i\vec{k}\cdot\vec{r}}\,d^3\vec{k} \tag{20}$$

$$= \int \Delta^2(k)\ e^{-i\vec{k}\cdot\vec{r}}\ \frac{dk}{k} \tag{21}$$

correlation function is Fourier transf of power spectrum $P(k)$
Q: why observationally useful?

example of general case: $P(k)$ “all you need know”
about density field for Gaussian fluctuations...

Power-Law Spectra

Consider a power-law power spectrum $P(k) \sim k^n$

- useful approximation over large k ranges
- inflation predicts initial conditions of this form
- recall $\Delta^2(k) \sim k^3 P(k) \sim k^{n+3}$
  homogeneity $\rightarrow n > -3$
  also must be cutoff at large k *Q: physical meaning?*

Note: this is only a first approximation
But we will see that the *true* power spectrum
is *not* a power law

- theory predicts deviations ("baryon acoustic oscillations")
- observations have begun to detect these

Rough meaning of n:
for a lengthscale $x \sim \lambda \sim 1/k$,
imagine "filtering" or "smoothing' density field over this scale
i.e., replace true density at each point with
density averaged over radius x

then for each lengthscale x
corresponding mean mass scale is $M \sim \rho_0 x^3 \sim x^3$
then $(\delta_{\rm rms})^2 \sim \int_0^{1/x} \Delta(k)\, dk/k \sim M^{-(n+3)/3}$
and so root-mean-square mass fluctuation is

$$\delta_{\rm rms} \sim M^{-(n+3)/6} \tag{22}$$

recall: for large k, $P(k) \sim k \to n = 1$
$\to \delta_{\rm rms} \sim M^{-2/3}$ drops for large masses:
approach homogeneity as $M \to \infty$

Correlation Function

if $P(k) \sim k^n$, then ξ also a power law:
$\xi(r) \sim r^{-(n+3)}$; for galaxies

$$\xi_{\rm gal}(r) \simeq \left(\frac{r}{5\ h^{-1}\ {\rm Mpc}}\right)^{-1.8} \tag{23}$$

where correlation length $r_{\rm corr} = 5\ h^{-1}$ Mpc
sets scale where ξ starts to become small
$\rightarrow$ typical structure size

note SDSS galaxy-galaxy ξ index gives $n \sim -1.2$
consistent with SDSS galaxy-galaxy $P(k)$ measurements
on the same scales (check!)

Filtered Density

Conceptually useful, and observationally practical
to imagine "filtering" the density field $\rho(\vec{x})$
over some lengthscale R, mass scale

$$M(R) = \rho_0 V(R) = 1.16 \times 10^{12} h^{-1} \left(\frac{R}{1\, h^{-1}\,\mathrm{Mpc}} \right)^3 M_\odot \qquad (24)$$

$\rightarrow$ gives "smoothed" field at this scale

To implement mathematically, introduce window function
weights the neighboring points; simplest is "top hat"

$$W(r;R) = \begin{cases} 1 & r \leq R \\ 0 & r > R \end{cases} \qquad (25)$$

using this, we have a "filtered variance"

$$\sigma^2(R) = \int d^3\vec{x}\ \delta(\vec{x})^2\ W(|\vec{x}|;R) \qquad (26)$$

$$= \frac{V}{(2\pi)^3} \int d^3\vec{k} P(k)\ W_k \simeq \Delta^2(k \sim 1/R) \qquad (27)$$

Scale of Nonlinearities Now

Key scale R: where $\sigma^2(R) = 1 \to$ linear/nonlinear boundary
empirically: near $R \sim 10$ Mpc
i.e., $M \sim 10^{15} M_\odot \to$ rich clusters!
$\to$ scale just becoming nonlinear today

key parameter set by convention: σ_8 a.k.a. "sigma-8"

$$\sigma_8^2 \equiv \sigma^2(8\ h^{-1}\,\mathrm{Mpc}) \simeq 0.8 \tag{28}$$

Gaussian Perturbations

So far: compared sizes of perturbations across different scales k
$\rightarrow$ via shape of $P(k) = |\delta_k|^2$

but can also ask: at one fixed scale k
what range of amplitudes δ_k appear?
i.e., sample Fourier amplitude δ_k over
different volumes $V \gg k^{-3}$
each a "realization" of true underlying cosmic sample
$\rightarrow$ what distribution results?

if Fourier mode amplitudes independent
and arise from causally disconnected regions
then central limit theorem ("law of averages")
$\rightarrow$ δ_k Gaussian distributed
$\rightarrow$ this is also prediction from inflation

i.e., for density field smoothed over size R
probability of finding fluctuation amplitude δ is

$$P(\delta) = \frac{1}{\sqrt{2\pi\sigma(R)}} e^{-\delta^2/2\sigma^2(R)} \quad (29)$$

implicitly require $|\delta| \ll 1$ *Q: why*

Observationally: holds as far as we can tell