Astro 596/496 PC

Lecture 34

April 16, 2010

Announcements:

- PF6 – last preflight! – due Monday noon
- **ICES** available online – please do it!

Last time: began theory of structure formation

$\rightarrow$ evolution of perturbations to a FLRW cosmology

Full technology: general relativistic perturbation theory

... and Boltzmann equation for evolution of

phase-space distribution function f for each cosmic species

Mostly good-enough technology: Newtonian cosmology

Summary of Jeans Analysis in Static Background

Sir James Jeans: Newtonian evolution of density perturbations of non-expanding, static homogeneous background fluid

$$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta \tag{1}$$

- wavelike solutions, e.g., $\delta_{\vec{k}} \propto e^{i(\vec{k}\cdot\vec{x} - \omega t)} \equiv D(t)\ \delta_0(\vec{x})$
- dispersion relation $\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \approx c_s^2 k^2 - 1/\tau_{\mathrm{frefall}}^2$
- critical scale: Jeans wavenumber $k_J = \sqrt{4\pi G \rho_0}/c_s$, Jeans wavelength $\lambda_J = 2\pi/k_J = c_s \sqrt{\pi/G\rho_0} \approx c_s \tau_{\mathrm{freefall}}$
- on large scales $k < k_J$: $\omega^2 < 0 \rightarrow D(t) = e^{+\omega t}$
  density contrast grows exponentially for large-scale modes!

⋆ **Jeans instability** or **gravitational instability**

Q: physical explanation? why only large-scale modes unstable?

Q: what if $\delta_0 < 0$?

Q: in expanding U, should grav instability be stronger or weaker?

Linear Theory I: Newtonian Analysis in Expanding U.

Recall: Newtonian analysis legal for small scales, weak gravity
$\longrightarrow$ okay for linear analysis inside Hubble length
apply to **matter-dominated U.**

Coordinate choices

Eulerian time-indep grid $\vec{x}$ fixed in physical space
expansion moves unperturbed fluid elts past as $\vec{x}(t) = a(t)\vec{r}$
Lagrangian coords $\vec{r}$ time-indep but expand in physical space
following fluid element: *locally* comoving
$\Rightarrow$ spatial gradients: $\nabla_{\vec{x}} = (1/a)\nabla_{\vec{r}}$

Unperturbed (zeroth order) eqs,
using $\rho_0 = \rho_0(t)$, $\vec{v}_0 = \frac{\dot{a}}{a}\vec{x} = \dot{a}\vec{r}$

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \vec{v}) = \dot{\rho_0} + \rho_0 \frac{\dot{a}}{a} \nabla_{\vec{x}} \cdot \vec{x} = 0 \tag{2}$$

$$\dot{\rho}_0 + 3\frac{\dot{a}}{a}\rho_0 = 0 \qquad \Rightarrow \rho_0 \propto a^{-3} \tag{3}$$

Poisson:

$$\nabla^2\Phi_0 = \frac{1}{x^2}\partial_x(x\partial_x\Phi_0) = 4\pi G\rho_0 \;\Rightarrow\; \Phi_0 = \frac{2\pi G\rho_0}{3}x^2 = \frac{2\pi G\rho_0}{3}a^2r^2$$

$$\nabla_{\vec{x}}\Phi_0 = \frac{4\pi G\rho_0}{3}\vec{x} \qquad \nabla_{\vec{r}}\Phi_0 = \frac{4\pi G\rho_0}{3}a\vec{r}$$

Euler

$$d(\dot{a}\vec{r})/dt = \ddot{a}\vec{r} = \frac{\ddot{a}}{a}\vec{x} = -\frac{4\pi G\rho_0}{3}\vec{x} \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho_0}{3} \tag{5}$$

Fried accel; with continuity $\rightarrow$ Friedmann

Zeroth order fluid equations $\rightarrow$ usual expanding U in non-rel approximation

Now add perturbations $\rho_1 = \rho_0 \delta$, $\vec{v}_1$, Φ_1

simplest to use comoving (Lagrangian) coords
follow observers in unperturbed Hubble flow
perturbation fluid elements $\vec{x}(t) = a(t)\vec{r}(t)$
peculiar fluid velocity $\vec{v}_1(t) = a(t)\vec{u}(t)$

plug in, keep only terms linear in perturbations ($\nabla = \nabla_{\vec{r}}$)
$\rightarrow$ *perturbation evolution to first (leading, linear) order*

$$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = -\frac{1}{a^2}\nabla\Phi_1 - \frac{1}{a}\frac{\nabla\delta p}{\rho_0} \tag{6}$$

$$\dot{\delta} = -\nabla \cdot \vec{u} \tag{7}$$

Note: if no pressure, density perturbations
still have $\dot{u} = -2Hu \rightarrow u \propto 1/a^2$, $v_1 \propto 1/a$
$\rightarrow$ pressureless fluid's peculiar vel redshifts same as free particle

Linearized Density Evolution

now look for plane-wave solutions $\leftrightarrow$ write as Fourier modes
e.g., $\delta(\vec{r}) \sim e^{-i\vec{k}\cdot\vec{r}}$, with $\vec{k}$ comoving wavenumber

$$\boxed{\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = \left(4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2}\right)\delta_k} \tag{8}$$

if no expansion ($a = 1, \dot{a} = 0$), recover Jeans solution

with expansion:

- Hubble "drag" $-2H\dot{\delta}$ opposes perturbation growth
- still critical Jeans scale: $k_J = \sqrt{4\pi G\rho_0 a^2/c_s^2}$
  expect oscillations on small scales, collapse on larger

Unstable Modes: Matter-Dominated U

Consider large scales $\gg \lambda_J$

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k \approx 4\pi G \rho_0 \delta_k \tag{9}$$

in Matter-dominated U: $8\pi G\rho/3 = H^2 = (2/3t)^{-2} = 4/9t^2$, so

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0 \tag{10}$$

eq homogeneous in $t \rightarrow$ try power law solution
trial $\delta \sim t^s$ works if $s(s-1) + 4s/3 - 2/3 = 0$
solutions $s = 2/3, -1$: growing and decaying modes

$$\delta_+(t) = \delta_+(t_i)\left(\frac{t}{t_i}\right)^{2/3} \;;\quad \delta_-(t) = \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-1} \tag{11}$$

- growing mode dominates
- Hubble friction: exponential collapse softened to power law

★ Note: solutions indep of k *Q: why a big deal?*

Linear Growth Factor

each unstable Fourier mode grows with time as

$$\delta_k(t) \propto D(t) \sim t^{2/3} \sim a \sim \eta^2_{\text{conform}} \tag{12}$$

indep of wavenumber k

- in k-space, all unstable modes grow by same factor and transform to real space, find
- one large scales (but still subhorizon)

$$\delta(t, \vec{x}_{\text{large}}) \simeq D(t)\delta(t_i, \vec{x}_{\text{large}}) \tag{13}$$

$\Rightarrow$ entire density contrast pattern grows with same amplification:

$\Rightarrow$ linear grow factor $D(t)$ applies to whole field

and thus, on subhorizon scales power spectrum evolves as

$P(k,t) = |\delta(k,t)^2| = D(t)^2 P_i(k) \sim a^2(t)\ P_i(k)$

$\Rightarrow$ RMS fluctuation $\sigma(R,t) \sim \Delta(k = 1/R) \sim a(t)\ \sigma_i(R) \sim \dfrac{\sigma_i(R)}{1+z}$

Applications to CMB: Naïve Inferences

before decoupling: pressure dominated by photons
$\rightarrow$ expect oscillations – and see them!
after decoupling: growing mode

CMB anisotropies are a snapshot
of perturbations at last scattering
can quantify level: $(\delta T/T)_{\rm ls} \sim 10^{-5}$ at $z_{\rm ls} \sim 1100$

But matter has $\rho \propto a^{-3} \propto T^3$, so $\delta\rho/\rho = 3\delta T/T$
$\rightarrow \delta(z = 1100) \sim 3 \times 10^{-5}$ at last scattering
So today, expect fluctuations of size

$$\delta_0 = \frac{D_0}{D_{\rm ls}}\delta_{\rm ls} = \frac{a_0}{a_{\rm ls}}\delta_{\rm ls} = (1 + z_{\rm ls})\delta_{\rm ls} \sim 0.05 \ll 1 \qquad (14)$$

Should still be very small–no nonlinear structures, such as us!
Q: obviously wrong–egregiously naïve! What's the flaw? What's the fix?

Director's Cut Extras

Linear Theory II: Sketch of Relativistic Treatment

see, e.g., Dodelson text, Liddle & Lyth Ch. 14

Recall limits of Newtonian treatment:

- only appropriate for scales $\lambda \ll d_H$: sub-horizon
- relativistic effects like time dilation absent or *ad hoc*

General Relativistic approach to cosmological perturbations

- as in Newtonian analysis, perturb density, velocity
  $\rightarrow$ this perturbs stress-energy
  schematically "$\delta T \approx \delta\rho + \delta P = \delta\rho + c_s^2 \delta\rho$"
- must therefore add small perturbations to metric:
  $g_{\mu\nu} = g_{\mu\nu}^{\mathsf{FRW}} + h_{\mu\nu}$
- these are related by Einstein's Equation
  $G_{\mu\nu} \approx$ "$\partial\partial g^{\mathsf{FRW}} + \partial\partial h$" $= 8\pi G_N T_{\mu\nu} \approx$ "$8\pi G_N(\rho + \delta\rho)$"

Metric Perturbations

Perturbations to metric tensor can be classified as:

- *scalar* – density perturbations couple to these
  these are most important
- *vector* – velocity perturbations couple to these
  these are least important (perturbations decay with time)
- *tensor* – source of gravity waves
  inflationary quantum perturbation excite these modes!

focus on *scalar* perturbations, which modify FRW metric thusly:

$$(ds^2)_{\text{perturbed}} = a(\eta)^2 \left[(1 + 2\,\Psi\,) d\eta^2 - (1 - 2\,\Phi\,) \delta_{ij} dx^i dx^j \right] \tag{15}$$

Coordinate freedom $\leftrightarrow$ "gauge" choice $\leftrightarrow$ spacetime "slicing"

$\Rightarrow$ here: *"conformal Newtonian gauge"*:

- $\Psi(\vec{x}, t), \Phi(\vec{x}, t)$ Schwarzchild-like forms if $a = 1, \dot{a} = 0$

Substitute perturbed metric into Einstein, keep only linear terms in Φ and Ψ, e.g., neglect Φ^2

use conformal time

and go to k-space

- $\nabla_\mu T^{\mu 0} \rightarrow$ "continuity"

$$\frac{d\delta}{d\eta} + ikv + 3\frac{d\Phi}{d\eta} = 0 \tag{16}$$

- $\nabla_\mu T^{\mu i} \rightarrow$ "Euler"

$$\frac{dv}{d\eta} + \frac{da/d\eta}{a} v + ik\Psi = \text{pressure sources} \tag{17}$$

- $G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} \rightarrow$ "Poisson"

$$k^2\Phi = -4\pi G a^2 \rho\delta \tag{18}$$

$$k^2(\Psi - \Phi) = -8\pi G a^2 \text{``}\langle P_x - P_y \rangle\text{''} \tag{19}$$

expect *anisotropic stress* small: $\langle P_x - P_y \rangle \ll \rho\delta \rightarrow \boxed{\Psi \approx \Phi}$

Recall: conformal time η gives particle horizon

On *sub-horizon* scales $\lambda \sim 1/k \ll \eta$:
relativistic treatment gives back Newtonian result
in fact: justifies our Newtonian treatment

On *super-horizon* scales $\lambda \sim 1/k \gg \eta$:
relativistic treatment still valid
$\rightarrow$ will use this to follow inflationary perturbations
through horizon crossing

Non-relativistic Cosmic Kinematics

gas particles have random thermal speeds, momenta
how are these affected by cosmic expansion?

Classical picture:
consider non-rel free* particle moving w.r.t. comoving frame
$\vec{\ell}_{\rm com}(t) \neq const$, and so $\vec{\ell}_{\rm phys} = a(t)\ell_{\rm com}(t)$:

$$
\begin{array}{rcccc}
\vec{v} = d\vec{\ell}_{\rm phys}/dt & = & \dot{a}(t)\ell_{\rm com}(t) & + & a(t)\dot{\ell}_{\rm com}(t) \\
 & = & H\vec{\ell}_{\rm phys} & + & \vec{v}_{\rm pec} \\
 & = & \text{Hubble flow} & + & \text{peculiar velocity}
\end{array}
$$

Note that peculiar velocity v is always w.r.t. the comoving frame–i.e., the particle speed compared to that of a stationary fundamental observer *at the same point*

*i.e., except for gravitation

consider a comoving observer at the origin, $\vec{x} = 0$
in time δt, a particle moves w.r.t. comov frame
physical dist $\delta \vec{x}_{\text{phys}} = \vec{v}_{\text{pec}} \delta t$

but due to Hubble flow, a comoving (fundamental) observer at $\delta \vec{x}_{\text{phys}}$ is moving away from the origin at speed $\vec{v}_{\text{com}} = H \delta \vec{x}_{\text{phys}}$

thus the new speed of the particle relative to its new comoving neighbor is given by the relative velocity
$\vec{v}'_{\text{pec}} = \vec{v}_{\text{pec}} - \vec{v}_{\text{com}}$
(where we used the non-rel velocity addition law)
and so the peculiar velocity *changes* by

$$\delta \vec{v}_{\text{pec}} = -H \delta \vec{x}_{\text{phys}} = -\frac{\dot{a}}{a} \vec{v}_{\text{pec}} \delta t = -\frac{\delta a}{a} \vec{v}_{\text{pec}} \tag{20}$$

Q: physical implications?

$\delta v_{\rm pec}/v_{\rm pec} = -\delta a/a \Rightarrow$ physical peculiar velocity $v_{\rm pec} \propto 1/a$:

- $\boxed{m v_{\rm non-rel} = p_{\rm non-rel} = p_0/a}$
- comoving peculiar velocity $d\ell_{\rm com}/dt \propto 1/a^2$
  slowdown w.r.t. comoving frame: velocity "decays"
  not a "cosmic drag" but rather kinematic effect
  due to struggle to overtake receding of cosmic milestones

Quantum picture:

recall for photons, $p_{\rm rel} = h/\lambda \sim 1/a$ (de Broglie)
but de Broglie holds for matter too: $p_{\rm non-rel} = h/\lambda_{deB} \sim 1/a$
$\Rightarrow$ again, $p_{\rm non-rel} = p_0/a$

true in general, now apply to thermal gas non-relativistic gas: Maxwell-Boltzmann

$$n = \frac{g}{(2\pi\hbar)^3} e^{-(mc^2-\mu)/kT} a^{-3} \int d^3p_0 \ e^{-p_0^2/2mka^2T}$$

if occupation number constant (particle conservation)
need $a^2 T(a) = T_0 = const$ and thus $T_{\rm non-rel} \propto 1/a^2$:

$$T_{\rm non-rel,decoupled} = \left(\frac{a_{\rm dec}}{a}\right)^2 T_{\rm decoupling} = \left(\frac{1+z}{1+z_{\rm dec}}\right)^2 T_{\rm decoupling}$$

evaluate for $z_{\rm dec} = z_{\rm ri}$: estimate

$$T_{\rm gas,today} \sim \frac{T_{\gamma,0}}{1+z_{\rm dec,gas}} \sim 6 \times 10^{-3} \ {\rm K} \tag{21}$$

Q: do the experiment...?

Q: what went wrong?

Inhomogeneities: The Spice of Life

So far: we have assumed perfect homogeneity!
If universe strictly homogeneous
indeed would cool to $T_{\rm gas} \ll T_0$

But happily, U. definitely inhomogeneous on small scales!
gravity amplifies density contrast *Q: why?*
"the rich get richer, the poor get poorer"

this allows for motion, condensation of matter
halo formation, mergers, shocks, star formation, quasars, ...
these overdense structures release energy
lead to diversity of cosmic matter and radiation today!

But how did we get the inhomogeneities?
And what set the primordial composition of baryons?
$\rightarrow$ events in the very early Universe...

Momentum Redshifting: Rigorously

the preceding heuristic arguments give the right result, but to obtain this rigorously requires General Relativity (if you haven't had GR yet, never mind)

in GR: a free particle's motion is a geodesic
so 4-momentum $p^\mu = m dx^\mu/ds = m(\gamma, \gamma\vec{v}) = (E, \vec{p})$ changes as

$$p^\alpha \nabla_\alpha p^\mu = p^\alpha \partial_\alpha p^\mu + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0 \tag{22}$$

and we see that the change in u is due to the connection term Γ, i.e., to curvature
$\rightarrow$ curvature tells matter how to move

note: homogeneity hugely simplifies: $p^\mu = p^\mu(t)$
so $\partial_\mu p = 0$ except for $\partial_t p = \dot{p}$

consider the $\mu = i \in (x, y, z)$ component of the geodesic eq

$$
\begin{aligned}
p^\alpha \partial_\alpha p^i + \Gamma^i_{\alpha\beta} p^\alpha p^\beta &= E\dot{p} + \Gamma^i_{\alpha\beta} p^\alpha p^\beta && (23) \\
&= 0 && (24)
\end{aligned}
$$

note that in FRW, if we write $ds^2 = dt^2 - h_{ij} dx^i dx^j$ where h_{ij} is the spatial metric, then nonzero $\Gamma^i_{\alpha\beta}$ are

$$\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i_j \tag{25}$$

where δ^i_j is the Kronecker delta (try it!)

We then have

$$E\dot{p}^i + \frac{\dot{a}}{a} E p^i = 0 \tag{26}$$

and thus

$$
\begin{aligned}
d\vec{p}/dt &= -\frac{\dot{a}}{a} \vec{p} && (27) \\
|\vec{p}| &\propto \frac{1}{a} && (28)
\end{aligned}
$$

Note that this result is completely general, i.e., works for all relativistic p, so

- in non-rel limit, $v \propto 1/a$: vel redshifts, and free particles eventually come to rest wrt the comoving background
- in ultra-rel limit, $v = p/E \approx c$, doesn't redshift, but since $E \approx p$, $E \propto 1/a$: energy redshifts

note classical derivation: didn't need Planck/de Broglie relation $p \propto 1/\lambda$ to show this (though that still works too)